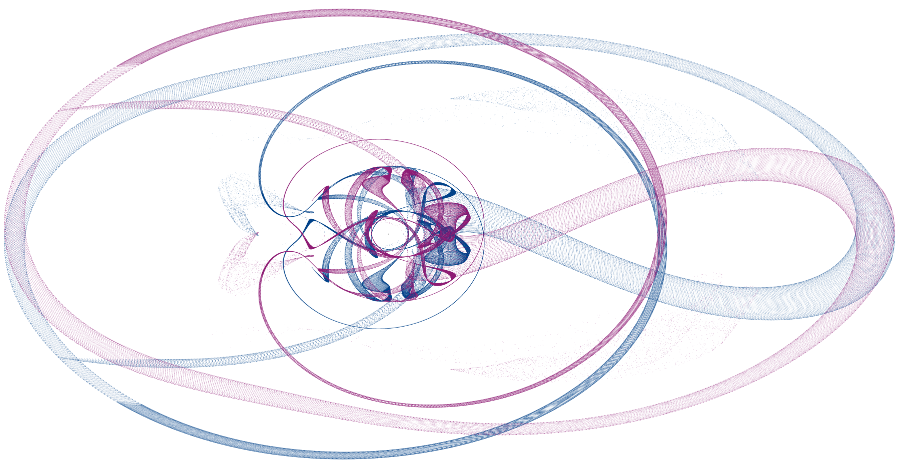

NoLineal 2025

14th international conference on

Nonlinear Mathematics and Physics

16 – 18 June 2025

Facultade de Matemáticas, USC, Spain

Organized by: Universidade de Santiago de Compostela,
Galician Centre for Mathematical Research and Technology

Welcome to NoLineal 2025 !

We are delighted to welcome you to NoLineal 2025, the *14^{th} International Conference on Nonlinear Mathematics and Physics*, taking place in Santiago de Compostela, Spain, from June 16th to 18th, 2025. This conference continues a well-established series that began in 1997, bringing together the nonlinear science community year after year.

As in previous editions, NoLineal 2025 will cover a wide range of topics, reflecting the diversity and richness of research in nonlinear science. Special emphasis will be placed on studies involving the modelling of physical phenomena through differential equations, although contributions from many other areas are also included in the program.

We are especially pleased to host the event in Santiago de Compostela, a welcoming and open city whose historic center was designated a UNESCO World Heritage Site in 1985. We hope you enjoy both the scientific sessions and the unique atmosphere of this beautiful city.

We would like to thank the Scientific Committee, our sponsors and all the participants for their valuable contributions. We wish you a pleasant stay in Santiago de Compostela and productive scientific and personal interactions during the conference!

NoLineal 2025 Organizing Committee

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Daily program

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10:50 – 11:20	Coffee Break		
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12:35 – 12:50	**Á Aguilar-Reyes** *Remarks on exponential attractors for a non-autonomous PDE with H^{-1}-valued forces*	**J. Ait Idir** *Mathematical modeling of phage-bacteria dynamics in vitro:quasineutral manifolds and bifurcations*	**M.C. Navarro** *Moist convective vortices: intensification by condensation*
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Monday afternoon, 16 June 2025

Aula Magna

15:00 – 16:00 **M. Ollé**
A naïve approach to splitting separatrices in the CP problem

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16:55 – 17:25 **Coffee Break**

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17:25 – 17:50	**P. F. Garrido** *Emerging navigation behaviors in a VR city: insights from a cognitive intervention in a twin cohort*	**J. Gimeno** *Tree-Based approaches for globalizing invariant manifolds in dynamical systems*	**I. Arán-Tapia** *Magnetohydrodynamic effects on the human endolymphatic fluid: a cause of dizziness in MRI*
17:50 – 18:15	**M. Saavedra** *Exploring spatiotemporal traffic dynamics in 25 cities via transfer entropy–driven causal networks*	**L. Martin Witkowski** *Computation and stability analysis of periodic orbits using finite differences, Fourier or Chebyshev spectral expansions in time*	**R. Dapena-García** *Study of particle dynamics in a constricted artery using the Lattice-Boltzmann method*

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Tuesday morning, 17 June 2025

Time	Aula 8 (J.Á. Cid)	Aula 9 (M. Gonchenko)	Aula 10 (M.C. Navarro)
	AULA MAGNA		
9:10 – 10:10	**J. Mawhin** *Two-ppoint boundary value problems and homoclinic solutions for systems of second order differential equations*		
10:15 – 10:40	**N. Khalil** *Quiescent states of a granular fluid in an external potential*	**S. Barbieri** *Existence and nonexistence on invariant curves of coin billiards*	**J.J. Mazo** *Plowing induced ripple formation in compliant materials*
10:40 – 11:05	**I. Gutiérrez-Sagredo** *Poisson-Lie groups and dynamics of nonlinear Hamiltonian dynamical systems*	**R. Capeáns** *Two-player game in a chaotic dynamical system*	**N. Bazarra** *Stability to double Timoshenko thermoelastic beam*
11:05 – 11:35	COFFEE BREAK		

Time	Aula 8 (J.Á. Cid)	Aula 9 (S. Barbieri)	Aula 10 (M. Ruíz-Villlarreal)
11:35 – 12:00	**D. Mellado Alcedo** *Stability of nonlinear Dirac solitons under the action of external potentials*	**M. Gonchenko** *Bifurcations of symmetric p:q resonant orbits*	**D. Martínez-Martínez** *Implicit Schwarz domain decomposition method with Legendre collocation for a Rayleigh-Bénard problem*
12:00 – 12:25	**A.P. Márquez** *New conserved quantities and modern symmetry analysis applied to a dissipative Westervelt's equation*	**J. Giné** *The expansion of the Poincaré map at monodromic singularities with inverse integrating factor*	**M. Picos** *Advanced Numerical Methods for DED AM Simulation: Multi-scale Domain Decomposition and Moving Meshes*
12:25 – 12:50	**A. Biasi** *Energy cascades and condensation via solvable Hamiltonian systems*	**A. Daza** *Chaotic dynamics of branched flows*	**H. Varela** *A stabilized numerical method for the Darcy-Forchheimer model*
13:00 – 15:00	LUNCH		

Tuesday afternoon, 17 June 2025

	Aula 8 (L. López-Somoza)	Aula 9 (A.J. Soares)	Aula 10 (J. Gimeno)
15:00 – 15:25	**D. Cao Labora** *The phenomenon of Borwein integrals from the perspective of Complex Analysis: classical and new identities*	**A.M. Portillo** *Modelling telomeres in stem cell populations*	**C. Arranz-Simón** *High-order rational methods for reaction-diffusion problems*
15:25 – 15:50	**A. Blasco** *Generalized quantum Zernike Hamiltonians: an algebraic approach to the spectrum*	**M. Stich** *Symmetry breaking and oscillations in simple models for polymerisation*	**P. P. Forrier** *A study of the Taylor coefficients of stiff ODEs*
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		Aula 9 (A. Fariña)	Aula 10 (I. Gómez-Bueno)
16:40 – 17:05		**J.M. López** *Lyapunov vectors as optimal paths in random media*	**C. Caballero-Cárdenas** *A semi-implicit fully exactly well-balanced relaxation scheme for the two-layer shallow water system*
17:05 – 17:30		**R. Gutiérrez** *Nonequilibrium criticality in the dynamics of synchronization in one dimension*	**F. Pla** *An alternating Schwarz domain decomposition method applied to the Rayleigh-Bénard convection problem*
20:00 – ...	Conference dinner (Hotel Palacio del Carmen)		

Wednesday morning, 18 June 2025

Aula Magna

9:10 – 10:10 **E. Gaburro**
High order structure preserving Lagrangian schemes for the solution of hyperbolic equations on moving Voronoi-like meshes with topology changes

	Aula 8 (J. Rodríguez-López)	Aula 9 (J. Campos)	Aula 10 (F. Pla)
10:15 – 10:40	**I. Baena Jiménez** *Information and dispersion measures for an oscillator of non-constant curvature*	**A. Paredes Vázquez** *Modeling Microbial Interactions with Generalized Lotka-Volterra Equations: Identification Challenges*	**I. Gómez-Bueno** *Well-balanced POD reduced-order models for nonlinear systems of PDEs*
10:40 – 11:05	**P. Cambeses-Franco** *Equations with reflection and piecewise constant dependence*	**A.J. Soares** *Analysis and simulations of a time-delayed model for tumor-immune system interactions*	**A. Crespo-Otero** *Global climatology of precipitation sources in atmospheric rivers using a Lagrangian approach*

11:05 – 11:35 **Coffee Break**

	Aula 8 (J. Rodríguez-López)	Aula 9 (A. Daza)	Aula 10 (M. Gómez-Mármol)
11:35 – 12:00	**L. López-Somoza** *Fourth order problem with functional perturbed clamped beam boundary conditions*	**J. Campos** *The shape of an epidemic wave*	**S. González-Pérez** *Using Lagrangian Coherent Structures to assess the impact of freshwater fronts in the Toconao pellet spill*
12:00 – 12:25	**S. Junquera** *The phenomenon of quenching in a system with non-local diffusion*	**L. Franco-Pérez** *The Spread of Two Serotypes of Vector-Borne Diseases in the Human Population*	**M. Agaoglou** *Uncertainty quantification and the method of Lagrangian descriptors*
12:25 – 12:50	**M. Yousfi** *Existence of solutions of nonlinear systems coupled to linear non-local boundary conditions*	**J. López-Pedrares** *Optimal control applied to viral competition*	**M. Ruiz-Villarreal** *A Lagrangian IBM model for modelling Early Life Stages of different fish and cephalopod species*

12:50 – 11:00 **Closing**

13:00 – 15:00 **Lunch**

Plenary Speakers

Application of stochastic control to some energy management problems

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The increasing complexity of energy systems, coupled with the uncertainty inherent in energy supply and demand, necessitates advanced decision-making frameworks. Stochastic control provides a powerful methodology to optimize energy management strategies in the presence of randomness. In this talk, we explore the application of stochastic control techniques to various energy-related problems, including optimal management of solar and wind power plants with storage, optimal consumption strategies, etc. We discuss continuous-time formulations, highlighting the role of Hamilton-Jacobi-Bellman (HJB) equations in deriving optimal control policies. Additionally, we consider cases where control actions are only permitted at discrete time instances and/or there exist state constraints requiring adaptations of classical dynamic programming approaches. Through analytical and numerical results, we demonstrate the effectiveness of stochastic control in improving energy efficiency and cost-effectiveness. The insights gained from these models contribute to the development of robust strategies for real-world energy management challenges.

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[2] World Energy Outlook 2024, Tech. report, *International Energy Agency*, Paris, 2024.

High order structure preserving Lagrangian schemes for the solution of hyperbolic equations on moving Voronoi-like meshes with topology changes

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In this talk, we present a *novel* family of high order accurate numerical schemes, of Finite Volume and Discontinuous Galerkin type, for the solution of hyperbolic partial differential equations (PDEs).

In particular, we work in the *direct Arbitrary-Lagrangian-Eulerian framework*, where each element of the mesh evolves following as close as possible the *local fluid flow*, so to reduce the dissipation at contact waves and moving interfaces and to respect Galilean and rotational invariance, but at the same time allowing also for the application of mesh optimization techniques. In addition, in our case, in order to give more freedom to the *mesh optimization*, we also allow for *topology changes* when regenerating our polygonal/polyhedral tessellations. Then, the previous mesh is connected with the new one by 3d/4d *space-time control volumes*, including *hole-like sliver* elements in correspondence of *topology changes*, over which we integrate a space-time divergence form of the original PDEs through a high order accurate ADER discontinuous Galerkin (DG) scheme.

Mass conservation and the respect of the GCL condition are guaranteed by construction thanks to the integration over closed control volumes, and robustness over shock discontinuities is ensured by the use of an a *posteriori* sub-cell finite volume (FV) limiter.

Moreover, our schemes are able to guarantee the exact preservation, up to machine precision, of equilibria and involution constraints: this allows us to obtain stable and robust simulations even for very complex physical models.

To prove the capabilities of our novel approaches, we will show a wide set of numerical results ranging from compressible *fluid-dynamics* over classical *magnetohydrodynamics* up to the study of the Einstein-Euler field equations of *general relativity*.

Keywords: hyperbolic partial differential equations; arbitrary Eulerian-Lagrangian methods-finite volume methods; discontinuous Galerkin schemes; structure preserving methods; topology changes.

Acknowledgements: E. Gaburro gratefully acknowledge the support received from the European Union with the ERC Starting Grant ALcHyMiA (No. 101114995).

Two-point boundary value problems and homoclinic solutions for systems of second order differential equations

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For systems of second order differential equations

$$u'' = f(t, u, u') \quad (t \in [0, T]),$$

the Hartman condition

$$|v|^2 + \langle u, f(t, u, v)\rangle \geq 0 \text{ when } |u| = R \text{ and } \langle v, u\rangle = 0$$

together with some Nagumo-type growth restriction on v is well known to ensure the existence of a solution satisfying the Dirichlet, Neumann, periodic or other two-point boundary conditions on $[0, T]$.

In this lecture we introduce some extensions of the classical Hartman type conditions under which the above mentioned boundary value problems are solvable.

We also give a very short proof of an estimate for the derivatives of solution due to Hartman.

The obtained results are applied to prove the existence of solutions to the homoclinic problem

$$u'' = f(t, u) \;\; (t \in \mathbb{R}), \quad \lim_{t \to \pm\infty} u(t) = 0,$$

Some open problems are mentioned.

Note: This is a joint work with Katarzyna Szymańska-Dębowska.

A naïve approach to splitting of separatrices in the CP problem

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We consider the Rydberg electron in a circularly polarized microwave field (the CP problem in short), whose dynamics is described by a 2 d.o.f. Hamiltonian, which is a perturbation of size $K > 0$ of the standard rotating Kepler problem. This problem has, in particular, a center-saddle equilibrium point L_1. We analyse the splitting phenomena of its associated invariant manifolds, for $K \to 0$. More precisely, we compute the distance between stable and unstable manifolds of L_1 by means of a semi-analytical approach plus numerical methods. Such distance turns out to be exponentially small. Also, we introduce a new family of *simpler* Hamiltonians, which we call **Toy CP** systems, that depend on two parameters. We discuss these systems and we show how, for suitable values of the parameters, a Toy CP problem already *mimics* the asymptotic behavior of the splitting of the CP problem.

Keywords: CP problem; invariant manifolds; numerical methods; Toy CP problems.

Acknowledgements: The Authors acknowledge financial support received from the MICIU and the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program.

Contributed talks

Uncertainty quantification and the method of Lagrangian descriptors

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In this talk, we will present a new uncertainty quantification measure appropriate for quantifying the performance of models in assessing for example the origin or source of a given observation. We have noticed that in a neighborhood of the observation this uncertainty measure is related to the invariant dynamical structures of the model such as hyperbolic trajectories and their stable and unstable manifolds.

Keywords: Uncertainty quantification; Ocean models; Invariant dynamical structures.

Acknowledgements: GGS and AMM acknowledge the support of a CSIC, Spain PIE project Ref. 202250E001. AMM, GGS and MA acknowledge the support from grant PID2021-123348OB-I00 funded by MCIN/AEI/ 10.13039/501100011033/ and by FEDER A way of making Europe. MA acknowledges the support from the grant CEX2019-000904-S and IJC2019-040168-I funded by: MCIN/AEI /10.13039/ 501100011033. AMM, GGS, and MA are active members of the CSIC Interdisciplinary Thematic Platforms TELEDETECT. SW gratefully acknowledges the support of the William R. Davis '68 Chair at the United States Naval Academy and the support of EPSRC, United Kingdom Grant No. EP/P021123/1.

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Remarks on exponential attractors for a non-autonomous PDE with H^{-1}-valued forces

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In this paper we ensure the existence of pullback exponential attractors for the process associated to a non-autonomous reaction-diffusion problem under minimal regularity assumptions, in fact with H^{-1}-valued time-dependent forces. The work is divided into two parts. In both cases, the well-posedness of the problem and suitable absorbing families allow to apply two different theorems of existence of pullback exponential attractors.

A first setting involves a translation bounded force $h \in L^2_b(\mathbf{R}; H^{-1}(\Omega))$ and certain nonlinearities. The existence of the pullback exponential attractor for this problem will follow the structure of the main result of [1].

The second block assumes a more general growth for the force. Namely it is allowed to grow exponentially fast in the past. This implies that the absorbing family also satisfies the same type of growth. This better generality allows us to apply the result on the existence of a pullback exponential attractor obtained in [2].

Keywords: Pullback exponential attractors, reaction-diffusion equations.

Acknowledgements: A.A.-R. is a doctoral student, fellow of PAIDI 2020 supported by Consejería de Transformación Económica, Industria, Conocimiento y Universidades de la Junta de Andalucía, reference contract PREDOC_00449. Research partially supported by the projects PID2021-122991NB-C21 (MICINN / FEDER, EU) and P18-FR-4509 (Junta de Andalucía / FEDER, EU).

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Mathematical modeling of phage-bacteria dynamics in vitro: quasineutral manifolds and bifurcations

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Antibiotic resistance represents a critical global health challenge. In this context alternative treatments need to be proposed. Phage therapy, which employs bacteriophages to target specific bacteria, emerges as a promising solution. In this study, we introduce a mathematical model that describes the interaction between a wildtype bacterial population, its phage-infected counterpart, and the emergence of resistant bacterial strains. The system is formulated through a set of nonlinear ordinary differential equations, incorporating key biological processes such as mutation, infection, and lysis.

We validate the model using experimental data through Bayesian inference and the Levenberg-Marquardt algorithm. A sensitivity analysis of the model is also carried out, proving its robustness. Stability analysis reveals the existence of multiple equilibria, including distinct lines of equilibria. We also verify experimentally the degeneracy in equilibria, providing additional support for the model's validity

Moreover, several bifurcations are found, highlighting conditions under which resistant bacteria persist or are eradicated. In particular, we also identify transcritical and Hopf bifurcation types, where the phage infection rate is used as a control parameter, illustrating possible oscillatory behaviors in bacterial and phage populations.

Our findings suggest that phage therapy alone is insufficient to eliminate resistant strains. However, modifying key parameters through external agents, such as antibiotic combination strategies, could drive resistance to extinction. Future research should explore these synergies and assess the perturbed model under extended experimental conditions.

Keywords: phage therapy; bacterial resistance; nonlinear dynamics; bifurcation analysis; mathematical modeling.

Acknowledgements: JAI acknowledges financial support received from the Spanish Ministry of Education Collaboration Fellowships. FD was supported by the Spanish Research Project PID2020-113052GB-100. JS acknowledge support from the MCIN/AEI through the María de Maeztu Program for Centers and Units of Excellence in *R&D* (CEX2020-001084-M) and the Generalitat de Catalunya CERCA Program.

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Magnetohydrodynamic effects on the human endolymphatic fluid: a cause of dizziness in MRI

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Magnetic resonance imaging (MRI) is a cornerstone of modern clinical diagnostics, with modern high-field MRI machines significantly increasing signal clarity and image resolution. However, the clinical utility of these systems is accompanied by reports of dizziness and nausea [1]. The underlying mechanisms of these symptoms remain only partially understood, with prevailing theories implicating a Lorentz force acting on the vestibular system in the inner ear, thereby generating a stimulus known as magnetic vestibular stimulation (MVS) [2, 3].

In this study, we develop a comprehensive mathematical model that integrates computational fluid dynamics (CFD), fluid–structure interaction (FSI) solvers, and magnetohydrodynamic (MHD) equations to simulate the biomechanical response of the vestibular system when subjected to a magnetic field. Leveraging high-resolution micro-computed tomography (μCT) data for anatomical precision [4], our model captures the dynamic behavior of the endolymphatic fluid under the influence of the magnetic field and its effect on vestibular receptors. The model has been clinically validated, successfully replicating the nystagmus (characteristic subtle eye movements) observed in humans at different head positions and magnetic field intensities [5].

Our findings substantiate the Lorentz force hypothesis as the principal mechanism driving MHD-induced vestibular stimulation in high-field MRI. The predictive capability of the model not only provides deeper physical and mathematical insights into the fluid dynamics of the inner ear, but also paves the way for enhancing MRI safety protocols, thereby enabling the implementation of higher-field MRI machines in current clinical practices.

Keywords: Magnetohydrodynamics (MHD); Computational Fluid Dynamics (CFD); Fluid-Structure Interaction (FSI); Magnetic Vestibular Stimulation (MVS); Endolymphatic Fluid.

Acknowledgements: This research is supported by the Spanish Ministerio de Economía y Competitividad and the European Regional Development Fund, through research grant PID2022-138322OB-100, funded byMCIN/AEI/10.13039/501100011033 and by “ERDF AWay of Making Europe.” It is also supported by Xunta de Galicia under research grant no. 2021-PG036, the

Instituto de Salud Carlos III (ISCIII) under grant number PI23/00248, and the National Institutes of Health (NIH) K23DC018302. The simulations were run in the Supercomputer Center of Galicia (CESGA).

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High-order rational methods for reaction-diffusion problems

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It is well known that standard Runge-Kutta methods of order p suffer from order reduction when applied to the time integration of parabolic or hyperbolic PDEs. This phenomenon causes the observed convergence order to be lower than p, significantly affecting numerical simulations in various applications. In this work, we address this issue by starting from the rational mappings of certain RK methods to construct a family of stable numerical schemes that retain order p, thus overcoming order reduction.

To this end, we consider the numerical time integration of abstract, initial boundary value problems of the form

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad t > t_0, \\ u(0) = u_0, \\ \partial u(t) = g(t), \quad t > t_0, \end{cases}$$

where $A : D(A) \subset X \to X$ is the infinitesimal operator of a $\mathcal{C}_0$ semigroup in a Banach space X, covering a broad range of problems of practical interest, including reaction-diffusion equations, Schrödinger-type models, wave propagation, and fluid dynamics.

The proposed numerical schemes achieve order p using only evaluations of the source term f, avoiding the computational cost of its derivatives. After an initial setup, each time step requires:

- a single evaluation of the source term f,
- an elliptic problem associated with $g(t)$,
- solving s linear systems, as in the original RK methods.

Additionally, we discuss how predictor-corrector techniques can be incorporated to further enhance accuracy and stability. These features make our methods particularly well-suited for simulations in fluid dynamics, wave propagation, and reaction-diffusion systems, where maintaining high accuracy and stability in long-time integration is crucial.

Keywords: Runge-Kutta methods; evolutionary PDEs; order reduction; reaction-diffusion problems.

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Information and dispersion measures for an oscillator of non-constant curvature

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In this talk, we present recent results on Rényi and Tsallis entropies [1] and local uncertainty measures [2] for the one-dimensional Darboux III oscillator.

The Darboux III oscillator is a λ-deformation of the Harmonic Oscillator. It is an exactly solvable quantum system that is defined on a space with non-constant curvature. Moreover, it can be interpreted as an oscillator with a position-dependent mass.

Information entropies in position and momentum spaces are functionals of the system's probability densities. They can be used to characterize the uncertainty or disorder of a quantum state, as well as to analyze the distribution of a quantum particle in these conjugate spaces.

Previous analytical results for the Shannon entropy of the Darboux III quantum oscillator were derived in [3]. In this talk, we discuss Rényi and Tsallis entropies—generalizations of Shannon entropy that depend on a parameter α—as well as dispersion measures in position and momentum space.

The need for a generalized virial theorem naturally emerges. Furthermore, different quantization schemes yield distinct wave functions in spaces with differently weighted scalar products. These schemes not only redefine the momentum operator but also lead to different interpretations of potential and kinetic energy. The limit of large curvature ($\lambda \to \infty$) is also explored.

Finally, results are compared in the limit $\lambda \to 0$ with those of the Harmonic Oscillator [4, 5].

Keywords: information entropy; Hermite polynomials; dispersion measures; Virial theorem; position-dependent mass.

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Existence and nonexistence of invariant curves of coin billiards

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In this presentation we consider the coin billiard introduced by M. Bialy. It is a modification of the classical billiard, obtained as the return map of a nonsmooth geodesic flow on a cylinder that has homeomorphic copies of a classical billiard on the top and on the bottom (a coin). The return dynamics is described by a map T of the annulus $\mathbb{A} = \mathbb{T} \times (0, \pi)$. We prove the following three main theorems: in two different scenarios (when the height of the coin is small, or when the coin is near-circular) there is a family of KAM curves close to, but not accumulating on, the boundary $\partial\mathbb{A}$; for any noncircular coin, if the height of the coin is sufficiently large, there is a neighbourhood of $\partial\mathbb{A}$ through which there passes no invariant essential curve; and the only coin billiard for which the phase space $\mathbb{A}$ is foliated by essential invariant curves is the circular one. These results provide partial answers to questions of Bialy. Finally, we describe the results of some numerical experiments on the elliptical coin billiard.

Keywords: billiards; twist maps; KAM Theory; geodesic flows.

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Stability to double Timoshenko thermoelastic beam

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In this talk, we will consider from both analytical and numerical points of view, a thermoelastic problem involving two elastic beams which are modeled by using the classical Timoshenko model. Double-beam systems are consisting of two parallel beams structure, connected together through an elastic layer which is generally considered as Winkler elastic. Nowadays, the double-beam structures have been widely used in many engineering fields, since they seem to be characterized by a better vibration absorption than a single beam, as well as lighter weight and higher strength and stiffness.

We analyze the well-posedness and the long time behavior and we present numerical results of a suitable initial boundary value problem related to a damped double Timoshenko thermoelastic beam model with thermal coupling on the bending moment under the Fourier law. The resulting problem is written in terms of the transverse displacements, the angles of rotation and the temperatures of the beams. The existence of a unique solution, as well as the exponential stability of the problem, are proved by using the theory of linear semigroups and energetic arguments. Then, we introduce a fully discrete approximation by using the finite element method and the implicit Euler scheme. The discrete stability of its solutions and a priori error estimates are shown, from which we can conclude the linear convergence of the approximations assuming an additional regularity on the continuous solution. Finally, numerical simulations are performed to demonstrate the numerical convergence and the behavior of the discrete energy.

Keywords: Timoshenko model; thermoelasticity; existence and uniqueness; energy decay; finite elements; a priori estimates; discrete stability.

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Energy cascades and condensation via solvable Hamiltonian systems

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One of the most fascinating features of nonlinear waves is their ability to form structures on vastly different scales. This property profoundly impacts the dynamics by giving rise to complex objects of diverse scales, such as condensates and solitons, and touches on fundamental problems in the theory of nonlinear evolution equations, such as singularity formation and loss of regularity. These processes are often explained through “cascades" in momentum space, where energy is transferred to either high-frequency or low-frequency modes. While these cascades are reasonably well understood in incoherent regimes, grounded in chaotic dynamics and wave turbulence theory, cascades driven by phase-sensitive dynamics (i.e., coherent regimes) remain challenging. Despite considerable advances over the past few decades, predicting when and how they occur in coherent regimes remains a difficult problem, and further research is needed to classify and understand the phenomena arising from them.

This talk will present recent progress on energy cascades in coherent/“deterministic” regimes. Two families of solvable Hamiltonian systems will be introduced, providing useful insights into the dynamical development of energy cascades and the formation of large- and small-scale structures from them. These problems are traditionally addressed through numerical methods, but in this case, I will show analytic solutions that describe the entire process. I will discuss how these solutions represent condensate formation, loss of regularity, spectral separation of conserved quantities, and the emergence of two types of singularities.

This talk is based on recent works [1, 2] and builds on previous studies [3, 4].

Keywords: Hamiltonian systems; energy cascades; condensation; solvability.

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Generalized quantum Zernike Hamiltonians: an algebraic approach to the spectrum

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Starting from the generalized classical Zernike Hamiltonian and its connections with deformed superintegrable oscillators defined within constant curvature spaces, we consider its quantum analog given by the following quantum Hamiltonian:

$$\hat{\mathcal{H}}_N = \hat{p}_1^2 + \hat{p}_2^2 + \sum_{k=1}^{N} \gamma_k (\hat{q}_1\hat{p}_1 + \hat{q}_2\hat{p}_2)^k,$$

with quantum canonical operators $\hat{q}_i$, $\hat{p}_i$ and arbitrary coefficients γ_k. This two-dimensional quantum model, besides the conservation of the angular momentum, exhibits higher-order integrals of motion within the enveloping algebra of the Heisenberg algebra. By means of a suitable combination of these integrals, we unravel a polynomial Higgs-type symmetry algebra that, through an appropriate nonlinear change of basis, gives rise to a deformed oscillator algebra.

This framework enables an algebraic determination of the possible energy spectra of the model for the cases $N = 2, 3, 4$, the case $N = 1$ being canonically equivalent to the harmonic oscillator.

Within this scenario, we propose two conjectures which generalize the results for any N and any value of the coefficients γ_k, that they are explicitly proven for $N = 5$. The principal point of the work hinges on the interpretation of the results as superintegrable perturbations of the original quantum Zernike system, corresponding to $N = 2$, which are also analysed and applied to the isotropic oscillator on the sphere, hyperbolic and Euclidean spaces.

Keywords: Zernike system; superintegrability; deformed oscillator algebras; polynomial symmetry algebras; eigenvalue spectra; momentum-dependent potentials; integrable perturbations; curved oscillators.

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A semi-implicit fully exactly well-balanced relaxation scheme for the two-layer shallow water system

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This work addresses the numerical approximation of the one-dimensional two-layer shallow water system, a depth-averaged model for stratified incompressible flows. It describes two immiscible fluid layers with different densities, coupled through non-conservative terms. These terms, together with the possible loss of hyperbolicity, pose significant challenges for numerical simulation.

In order to obtain efficient schemes to approximate the solution of this system, we follow the strategy proposed in [1] and develop semi-implicit numerical methods based on splitting and relaxation strategies. The splitting separates the acoustic and transport components, and the semi-implicit treatment of the acoustic step allows for larger time steps than in fully explicit schemes, significantly improving computational efficiency while maintaining stability. The method is designed to be fully well-balanced, meaning it exactly preserves every possible steady state, including the lake-at-rest and more general equilibrium configurations. The proposed approach is extended to second-order accuracy in both time and space. A set of numerical experiments is presented to validate the performance, accuracy, and efficiency of the method.

Keywords: semi-implicit schemes; fully exactly well-balanced schemes; two-layer system; splitting strategy; relaxation schemes.

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Equations with reflection and piecewise constant dependence

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In the expansive and continuously evolving field of mathematics, differential equations play a crucial role in modeling and understanding phenomena across various disciplines. However, the challenges involved in modeling complex systems, where interactions are not strictly local or changes occur in discrete intervals, have led to the development of novel classes of differential equations. Among these, nonlocal differential equations and differential equations with piecewise constant arguments are particularly notable.

Nonlocal equations are widely used in fields such as quantum physics or medicine, where they are especially useful in modeling the dynamics of infectious disease transmission. Our focus is on a particular type of nonlocal equation: equations with involution. These are characterized by the property that the function composed with itself is the identity. The study of these equations began with Silberstein in 1940, who analyzed the solution to the equation $v'(t) = v(\frac{1}{t})$ [4], and, since then, many authors have contributed to this area of research [2].

On the other hand, differential equations with piecewise constant arguments are important for modeling situations where the dependent variables or their derivatives are evaluated at discrete points in time and/or space. These equations are widely used in fields such as control engineering, biomedicine, and economics. Their study began in the early 1980s and has since been extensively treated in the literature [3].

The objective of our work is to study equations with involution and piecewise constant arguments together [1]. In particular, we focus on equations of the form

$$L_n v(t) + m\, v(-t) + M\, v([t]) = \sigma(t), \text{ a.e. } t \in J, \quad V_i(v) = h_i, i = 1, \dots, n, \tag{1}$$

where

$$V_i(v) \equiv \sum_{j=0}^{n-1} \left(\gamma_j^i v^{(j)}(-T) + \eta_j^i v^{(j)}(T) \right), \quad i = 1, \dots n,$$

and

$$L_n v(t) \equiv v^{(n)}(t) + a_1(t) v^{(n-1)}(t) + \cdots + a_{n-1}(t) v'(t) + a_n(t) v(t), \quad t \in J := [-T, T],$$

being γ_j^i, η_j^i, and h_i real constants for all $i = 1, \dots, n$ and $j = 0, \dots, n-1$, and σ, $a_k \in \mathcal{L}^1(J)$ for all $k = 1, \dots, n$.

We approach this problem using the theory of Green's functions and analyzing their properties. First, we derive the explicit expression for the Green's function $H_{m,M}$ of Problem (1), obtaining that

$$H_{m,M}(t,s) = G_m(t,s) - M \left[\sum_{j=-[T]}^{[T]} \sum_{i=[-T]}^{T} G_m(j,s) \int_{-T}^{T} G_m(t,r) \alpha_{ij}(r) \mathrm{d}r \right],$$

where G_m is the Green's function of Problem (1) with $M = 0$, which can be deduced following [2] and α_{ij} are the elements of the inverse of a matrix that depends on G_m.

Next, we characterize this function by studying its properties and delineating the set of parameters involved in the analyzed equations for which the Green's function exhibits a constant sign. To do this, we stablish the following relationship between the Green's functions H_{m_0,M_0} and H_{m_1,M_1}, related to Problem (1) with parameters (m_0, M_0) and (m_1, M_1) respectively:

$$H_{m_0,M_0} = (m_1 - m_0)\int_{-T}^{T} H_{m_1,M_1}(t,r)H_{m_0,M_0}(-r,s)\mathrm{d}r \\ + (M_1 - M_0)\int_{-T}^{T} H_{m_1,M_1}(t,r)H_{m_0,M_0}([r],s)\mathrm{d}r + H_{m_1,M_1}(t,s).$$

Thus, we develop a new method to characterize the regions of constant sign of the Green's function related to Problem (1).

With all this in hand, and knowing where the Green's function has a constant sign, we demonstrate the existence of solutions to the associated nonlinear problems by applying various results derived from fixed-point theory. In particular, we can apply the monotone method, Krasnosel'skii fixed-point theorem, the Leray-Schauder method, or the Krasnosels'kii-Zabreiko method.

The aforementioned methodology is specifically applied to the first-order linear problem with periodic conditions $v'(t) + m\,v(-t) + M\,v([t]) = h(t)$ for $t \in [-T, T]$, proving several existence results for the associated nonlinear problem and precisely delineating the region where the Green's function $H_{m,M}$ has a constant sign.

Keywords: Green's function; equation with reflection: piecewise constant arguments; constant sign solutions; explicit expression.

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The shape of an epidemic wave

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The recent Covid-19 epidemic has made us see what an epidemic looks like. The most basic way to simulate these waves is to use the model described in [2]. There, a compartmental model was presented based on three groups of individuals, *susceptible, infected* and *recovered*. However, it has been very common to only present infected individuals profile or at most recovered individuals.

A method is proposed to understand biological parameters that only uses the profiles of infected individuals or at most recovered individuals, by using new parameters that are: epidemic severity, wave asymmetry and fraction of endemicity. These results follows from [1].

Keywords: epidemic severity; wave asymmetry; fraction of endemicity.

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The phenomenon of Borwein integrals from the perspective of Complex Analysis: classical and new identities

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Borwein integrals were first described by David Borwein and Jonathan Borwein in 2001. They form a simple family of integrals involving the cardinal sine function, denoted by $\operatorname{sinc}(x) = \sin x/x$, such that the first few integrals are equal to π until, unexpectedly, this pattern breaks. More specifically, we have:

$$\begin{aligned}
I_1 &= \int_{-\infty}^{\infty} \operatorname{sinc}(x)\, dx = \pi, \\
I_2 &= \int_{-\infty}^{\infty} \operatorname{sinc}(x)\cdot\operatorname{sinc}(3x)\, dx = \pi, \\
I_3 &= \int_{-\infty}^{\infty} \operatorname{sinc}(x)\cdot\operatorname{sinc}(3x)\cdot\operatorname{sinc}(5x)\, dx = \pi, \\
&\ \vdots \quad \text{and this predictable pattern continues until} \\
I_8 &= \int_{-\infty}^{\infty} \prod_{j=1}^{8} \operatorname{sinc}((2j-1)x)\, dx \neq \pi.
\end{aligned}$$

In fact, the correct value of I_8 is approximately $\pi - 2.31 \times 10^{-11}$. The classical explanation for this unexpected phenomenon relies on Fourier analysis techniques and was originally presented by Borwein father and son in [1].

In this contribution, we demonstrate how this result, along with similar ones, can be derived using elementary complex analysis tools, specifically residue theory. Furthermore, this complex analysis perspective allows us to go beyond the classical result and its associated identities. In particular, we describe a generalization that considers the case of three dominant frequencies, which, as a straightforward consequence, leads to numerous new formulas involving "sinc" and π. The main content of this contribution is based on [2].

Keywords: Borwein integrals; residue theory; complex analysis; cardinal sine function.

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Two-player game in a chaotic dynamical system

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Imagine two players pulling an object. Player A wants to pull it to its side, while Player B pulls toward its own. Without any other factors involved, the stronger player would simply win by pulling harder.

However, the game becomes much more interesting when the object naturally moves according to a rule given by a function f. Now, both players must not only worry about each other's pulling strength, but also need to consider how the object naturally moves. This becomes especially tricky when the object's movement described by f is chaotic, meaning that even tiny changes in how they pull can lead to completely different and unpredictable results as illustrated in Fig. 1.

This presentation explores such competitive scenarios in chaotic systems, where two players apply limited control forces to achieve opposing goals. The challenge arises from three interacting factors: the unpredictable nature of chaos, the bounded control available to each player, and the strategic competition between adversaries.

Our analysis focuses on determining which initial conditions favor each player, identifying regions of phase space where a player can guarantee victory regardless of opponent actions.

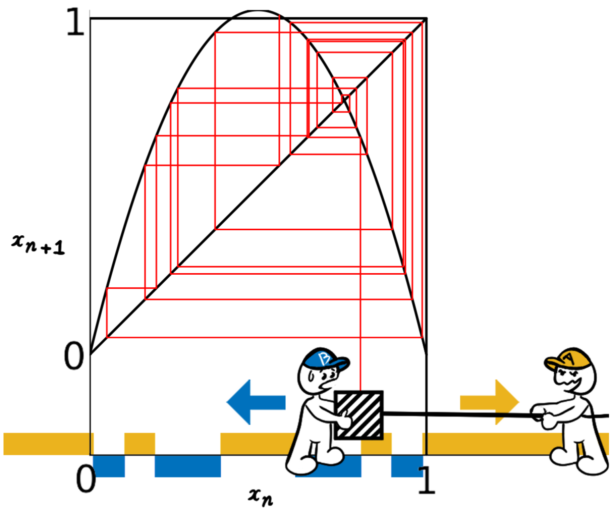

Figure 1. Competition for control in the chaotic logistic map, where $f(x) = \mu x(1-x)$ with $\mu = 4.1$. The blue player aims to keep the trajectory within the interval $[0, 1]$, while the yellow player attempts to force it to escape. The system evolves according to $x_{n+1} = f(x_n) + u_n^A + u_n^B$, with u_n^A and u_n^B representing the bounded control inputs from each player. The yellow set guarantees victory for the yellow player, while the blue set guarantees victory for the blue player.

Keywords: transient chaos; control theory; game theory.

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Dispersal effect in a periodic discrete two-patch model

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Understanding the evolution of difference equations modelling the dispersion of populations in a landscape consisting of several patches it is an interesting problem, both from the mathematical point of view and from applications like conservation management and the design of ecological corridors, [1, 2, 3].

In particular, in [2] the authors considered an autonomous discrete model, with two patches and a constant dispersal rate $\delta \in [0, 1]$, and they gave a complete analysis of the responses of the asymptotic total population size when δ increases from $\delta = 0$ (isolated patches) to $\delta = 1$ (exchange of populations). Our aim in this talk is to present a nonautonomous generalization of this model with k-periodic parameters and provide, under rather general assumptions, the existence and global atractivity of a unique positive k-periodic solution. Moreover, we extend to this framework some known results for the autonomous model and we also obtain new scenarios in the response of the average asymptotic total population size to dispersion.

Keywords: Population dynamics; Two-patch model; Periodic solution; Total population size.

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Global climatology of precipitation sources in atmospheric rivers using a Lagrangian approach

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Atmospheric rivers (ARs) are narrow bands of concentrated moisture in the atmosphere, usually located in subtropical and mid-latitude oceanic regions. Their formation, persistence, dynamics, and impacts involve complex non-linear interactions, which require the use of dynamical system tools and statistical descriptions. ARs play a crucial role in the climate of western coastal regions, as they contribute a large part of the annual precipitation recorded in these areas. Moreover, they can trigger extreme rainfall when the vast amounts of water vapor they transport are abruptly forced upwards. As a result, ARs have been the focus of extensive research in recent years, and one of their most debated properties is the origin of the moisture in them. While some studies have identified moisture sources using various diagnostic tools for specific AR cases, the relative dominance of tropical versus extratropical contributions remains uncertain.

To address this gap, a global climatology of moisture sources for precipitation in ARs is needed. There are a number of moisture source diagnostics that can be used for this purpose. One of the most common approaches is the use of Lagrangian tools that compute the backwards-in-time trajectories of air parcels and their moisture content (along with other atmospheric properties), effectively modeling the transport process as a dynamical system that leads to a source-destination network. We follow this approach by using the Lagrangian model FLEXPART combined with an implementation of the Dirmeyer and Brubaker (1999) methodology [1], which we previously validated against the WRF with Water Vapor Tracers (WRF-WVTs) model [2, 4]. Since WRF-WVTs is considered to be one of the most accurate moisture tracking tools, this validation demonstrates the effectiveness of the Lagrangian approach. Moreover, its computational efficiency allows us to simulate air particle trajectories within a wide range of ARs, enabling a detailed characterization of moisture sources. Identification of ARs and associated precipitation events has been done with a deep-learning neural network based on CGNet [3].

A statistical description in terms of a preliminary climatology reveals a wide diversity of moisture sources, spanning both oceanic and continental regions, with substantial variability across different AR cases. This diversity is partly explained by the geographic location where the AR occurs. As expected, ARs are effective in transporting water from more remote locations compared to non-AR precipitation events. Finally, our findings suggest a less significant role of tropical moisture than previously thought, underscoring the complexity of moisture uptake in ARs.

Keywords: atmospheric river; moisture transport; Lagrangian model.

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Study of particle dynamics in a constricted artery using the Lattice-Boltzmann method

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Cardiovascular diseases are a leading cause of death globally. Among them, some are linked to stenosis, which is an abnormal narrowing of blood vessels, as well as other factors. Smart drug delivery systems based on micro- and nanoparticles are a promising method to offer non/minimal-invasive therapeutic mechanisms. Several types of shear stress-sensitive drug vehicles have been proposed, where drug release is triggered by local shear changes [1]. Recently, there has been growing interest in shear-activated particle clusters for the treatment of stenosis [2]. However, to the best of our knowledge, no existing drug carriers are specifically guided to the vascular lesion by flow characteristics and changes in wall shear stress, enabling both localized internalization in diseased cells and sustained therapeutic release. Understanding the factors influencing margination, particularly in complex vascular geometries like stenotic arteries, is vital for advancing biomedical applications.

In this study, we investigate the behaviour of particles of varying shapes and sizes, including red blood cells (RBCs), under stenotic flow conditions in an artery using the Lattice-Boltzmann Method (LBM) [3, 4]. Rooted in the mesoscale, the LBM models the fluid as a collection of particle distribution functions, and simulates their evolution over time. In recent years, LBMs have evolved enough to compete with classical tools in solving complex fluid flows problems, including those with particle migration.

Here we model blood as a Newtonian fluid, while particles are tracked from a Lagrangian perspective to capture the drag, lift, and torque forces acting on them. A stenosed artery geometry (see Fig. 2) is employed to replicate the hemodynamic disturbances characteristic of such regions, including recirculation zones and high shear rates [5]. Margination dynamics are analyzed by tracking particle trajectories and distributions in response to these flow variations. Particular attention is given to how particle properties such as deformability, size, and shape, affect their lateral migration and adhesion probabilities.

Preliminary results reveal that non-circular particles exhibit faster and more pronounced margination compared to circular ones [6], with deformable particles, such as RBCs, showing distinct interaction patterns. These findings suggest that tailoring particle geometry and mechanical properties could enhance the efficiency of targeted drug delivery in stenotic arteries. Further insights from this study could inform the design of next-generation nanoparticles for precision medicine, improving the effectiveness of drug carriers in regions of vascular pathology.

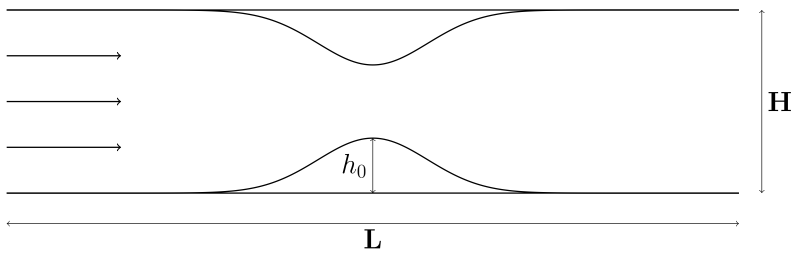

Figure 2. Two-dimensional representation of the simulation setup..

Keywords: Lattice-Boltzmann method; Particle dynamics; Blood flow; Constricted flow.

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Chaotic dynamics of branched flow

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Tsunamis, ants, lasers, and electrons share a common feature: as they travel, they can generate a striking pattern known as branched flow [1]. These transient structures illustrated in Fig. 3, efficiently transport energy over unexpectedly long distances, with tsunamis being perhaps the most dramatic example of their powerful consequences. The conditions that give rise to branched flow naturally place it within the semiclassical regime, allowing its dynamics to be captured through different approximations including ray-tracing and a conservative map. Despite its ubiquity, branched flow has only recently been recognized, and its fundamental mechanisms remain an open field of study.

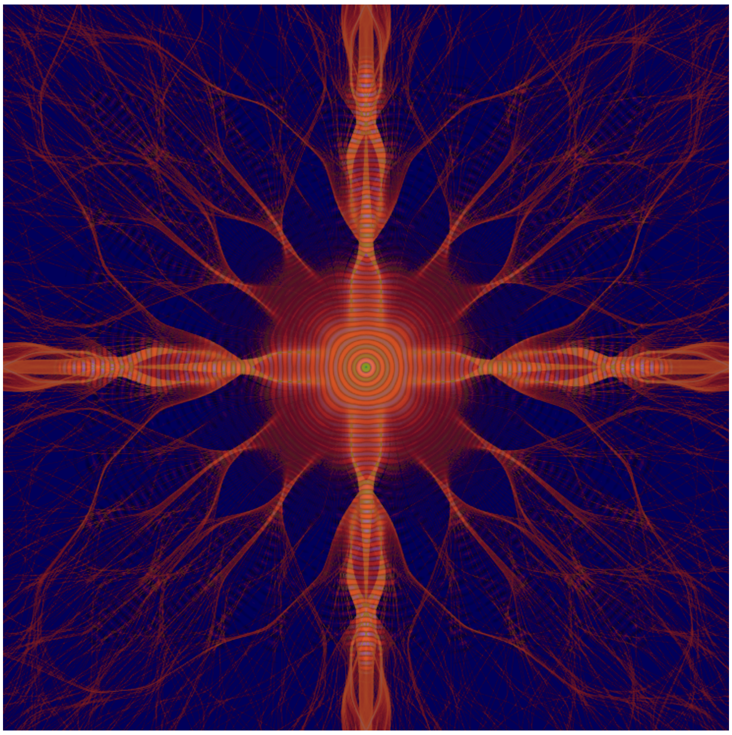

Figure 3. Branched flow pattern in a periodic medium. The wave dynamics and the density of classical trajectories are superimposed in this picture, revealing the semiclassical nature of branched flow..

Most of the current literature focuses on the appearance of branched flow in structureless media. Nevertheless, branched flow can also happen in periodic superlattices [2], opening a whole new sphere of possible applications including twistronics and photonics. Moreover, in periodic potentials, branched flow presents infinitely stable branches called superwires [3]. Chaotic dynamics plays a fundamental role in the creation and destruction of both branched flow and superwires [4], hinting at deep connections with the celebrated standard map. This link also enables the possibility of shaping branched flow through chaos control techniques.

Keywords: branched flow; superwire; standard map; wave propagation; semiclassical.

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Finite-Size Effects in the Scaling of A + B → C Reaction Fronts

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Reaction-diffusion (RD) fronts play a key role in various physical, chemical, and biological processes, such as disease spreading, finances or geological applications such as carbon dioxide sequestration. In this study, we investigate how finite geometry affects the evolution of A + B → C reaction fronts when the reactants are initially segregated in space.

Previous studies have shown that in infinite systems, key front parameters such as position x_f, width w, and maximum production rate R follow well-defined power-law scaling behaviors over time [1]. Here, we extend this analysis to finite geometries, identifying two additional time regimes that emerge due to spatial constraints and system properties. We demonstrate that, depending on the initial concentration ratio and diffusion coefficients, the front may either reach a stationary position and exhibit transitional dynamics with exponential scaling. We validate these findings using numerical simulations [2].

These results highlight the significant impact of geometry on RD front dynamics and may have applications in a variety of areas depending on the interpretation of A, B, and C.

Keywords: RD Fronts; Finite size effects; Scalings.

Acknowledgements: The Authors acknowledge financial support received from Prodex and the Belgian Science Policy Office (BELSPO).

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Compactification techniques for analyzing asymptotic behavior in partial differential equations

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In this presentation, we explore an approach to examining the asymptotic behavior of solutions to partial differential equations (PDEs) through the compactification of domains. By introducing new functional spaces tailored to these compactified domains, we establish a version of the Ascoli-Arzelà Theorem applicable in this context. Furthermore, we develop fixed point index results essential for proving the existence and multiplicity of solutions within these spaces. This methodology provides a unified framework for analyzing both the solutions of PDEs and their asymptotic properties.

This talk is based on joint work [1] with Prof. Lucía López-Somoza (University of Santiago de Compostela).

Keywords: Topological methods for PDE; compactification; existence; Ascoli Theorem.

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The Spread of Two Serotypes of Vector-Borne Diseases in the Human Population

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Mosquitoes are one of the most significant vectors for transmitting infectious diseases to humans and animals. They are responsible for spreading a variety of dangerous diseases, including Malaria, Dengue fever, Chikungunya, and Zika virus, which affect millions of people worldwide. Mosquitoes transmit these pathogens through their bites, with different species carrying distinct viruses, bacteria, and parasites. Malaria alone causes over 200 million cases annually, while Dengue fever, which is particularly prevalent in tropical and subtropical regions, is responsible for hundreds of thousands of cases and deaths each year. The ability of mosquitoes to spread these diseases makes them a major public health concern globally.

Dengue fever is caused by four distinct strains of the dengue virus (DENV-1, DENV-2, DENV-3, and DENV-4). An individual infected with one strain can be at risk of contracting another strain after recovery. Although a vaccine for dengue fever exists, it is only recommended for individuals who have been previously infected with one of the serotypes. For those who have not had a previous infection, a subsequent dengue infection can be more severe.

To better understand viral dynamics and the key factors driving their spread among humans, we are developing a dynamic system. This system incorporates an SIR model for the human population, accounting for vital dynamics, alongside a compartmental model for the mosquito population. The mosquito model includes factors such as the Allee effect, logistic growth, and density-dependent mortality rates [1]. The two models are interconnected, and we specifically focus on the infected human and mosquito populations, considering the two dominant serotypes of the virus in a particular region, as well as vertical transmission within the mosquito population.

Keywords: vector-borne diseases; bifurcation; sir model.

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Quantum-like behavior of an active particle

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A macroscopic, self-propelled wave-particle entity (WPE) that emerges as a walking droplet on the surface of a vibrating liquid bath (Fig. 4) exhibits several hydrodynamic quantum analogs [1]. We explore the rich dynamical and quantum-like features emerging in a model of an idealized one-dimensional WPE in a double-well potential [2]. The integro-differential equation of motion for the WPE transforms to a four dimensional Lorenz system [3]. We observe the analog of quantized eigenstates in the form of multistability with coexisting limit cycles that arise via Hopf bifurcation. These states show narrow as well as wide energy level splitting. Tunneling-like behavior is also observed where the WPE erratically transitions between the two wells of the potential [2]. We rationalize this phenomena in terms of crisis-induced intermittency [4]. Finally, we discover a Cantorian fractal structure in the escape time distribution of the particle from a well based on initial conditions, indicating unpredictability of this tunneling-like behavior at all scales. The chaotic intermittent dynamics leads to wave-like emergent features in the probability distribution of the particle's position, which are qualitatively similar to its quantum counterpart.

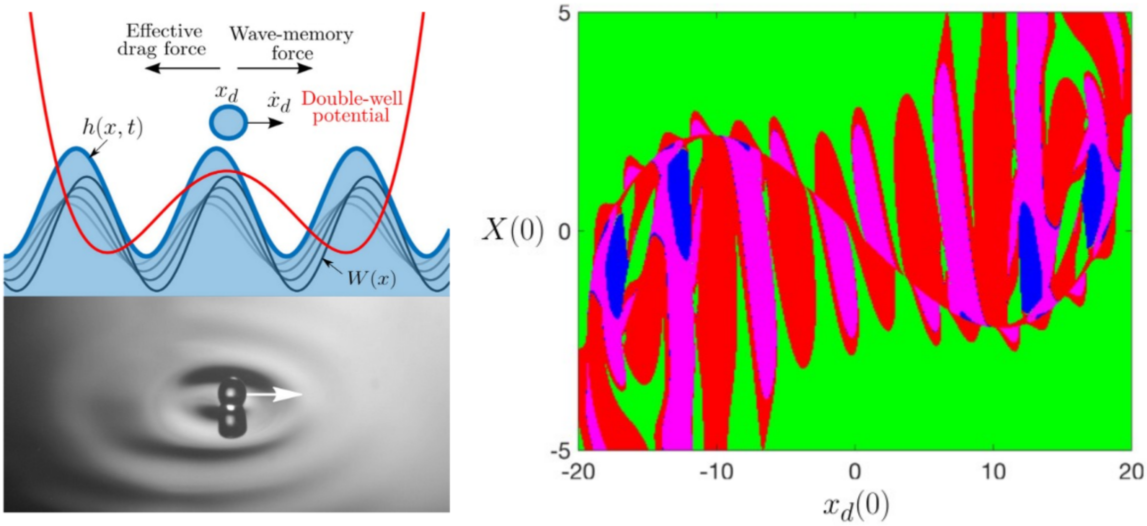

Figure 4. Left: A bouncing silicon droplet on a vibrating bath under the action of an external potential, showing how De Broglie's pilot-wave dynamics arises in classical hydrodynamics (Image courtesy of Rahil Valani, Oxford). Right: Basins of attraction of the four energy levels that arise via Hopf bifurcation in terms of coexiting limit cycles. The colors correspond to quantized orbits with increasing Lyapunov energies..

Keywords: Hydrodynamic quantum analogs; Time-delay; Lorenz system; Hopf bifurcation; Orbit quantization; Intermittency.

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Emerging navigation behaviors in a VR city: insights from a cognitive intervention in a twin cohort

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Navigation is the ability of a person or an animal to orient themselves in space in order to move from one point to another. It is a complex behavior that involves several processes such as memory, sensory perception, cognitive mapping, and decision-making. Navigation difficulties have been associated with aging and identified as one of the first early symptoms in diseases such as Alzheimer's [1, 2, 3]. Cognitive training has shown memory improvements in numerous studies, but its use for better understanding strategies in human navigation and potential changes with time and training is less explored. This study aims to investigate navigation behaviors and strategies using cognitive training in a virtual environment.

A cognitive experiment intervention has been designed in a unique 3D virtual city. The city covers an equivalent extension of 5 km^2 and is divided into 10 different levels. These levels are unlocked every time a participant reaches a specific goal, increasing the complexity of the city streets with training. A session initially consists of looking for different objects placed in specific parts of the city. The targets are given to the participants one by one in a randomized set for every session. Additionally, these targets are always located in the same coordinates, meaning that every participant can become familiar with them before they are actual targets. To complement the VR training, a laptop version of the city was created for home sessions. The experiment is stated to last for 10 weeks, and every person is asked to train several times a week, both in the VR laboratory and at home.

206 participants from an adult twin cohort entered the study. Half of the sample were classified as trainers and 100 completed more than 2 h in the VR city, of which 28 monozygotic and 15 dizygotic full twin pairs reached this threshold. Participants navigate in the city wearing a Virtual Reality (VR) mask while seated on a bicycle on a bike roller. During each session, their coordinates in the city are recorded every 0.1 seconds, as well as the angles or their head movement (pitch, roll, and yaw). Derived magnitudes, as displacement and head angular velocities, can be calculated afterwards.

The different strategies used to explore the city and navigate through it are detected using an unsupervised approach. Given the movement of the head and the velocity on the bike, our aim is to detect and identify the various behaviors used by each person throughout the training time and follow their evolution. Navigation strategies are identified using Unsupervised Multivariate Time Series Segmentation and Clustering with two different approaches: using Hidden Markov Models (HMMs) and Random Markov Fields (RMFs) applying the Toeplitz Inverse Covariance-Based Clustering [4].

The analysis shows a minimum of 5 clusters that are repeated along all the examined sessions and participants. A participant's behavior during a session is composed by a collection of the detected clusters. The strategy they use depends on their familiarity with the neighborhood, the distance towards the following target, and personal variables such as age. The combination of the two unsupervised approaches lets us detect which clusters are closer to the common allocentric and egocentric navigation strategies.

Keywords: Virtual Reality; Navigation; Time Series Clustering; Machine Learning; Cognition.

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Tree-based approaches for globalizing invariant manifolds in dynamical systems

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Understanding the global structure of invariant manifolds is crucial in the study of dynamical systems, with applications spanning celestial mechanics, fluid dynamics, and control theory. However, traditional numerical methods often struggle with scalability and accuracy when extending local manifold approximations to a global representation. In this talk, we present an approach that leverages tree-based data structures, widely used in computer graphics, to efficiently organize, propagate, and identify intersections of manifolds. By integrating adaptive Chebyshev function approximation techniques, we construct smooth and computationally efficient representations of invariant manifolds with controlled error estimation. Additionally, we employ heuristic-based strategies to further enhance numerical stability and computational efficiency. We illustrate the effectiveness of our method through examples in both low- and high-dimensional dynamical systems, demonstrating its potential for advancing global manifold analysis. This framework can open new avenues for scalable and precise studies of complex dynamical behaviors.

Keywords: Globalization of Manifolds; Tree-Based Data Structures; Chebyshev Approximation; Manifold Intersections.

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The expansion of the Poincaré map at monodromic singularities with inverse integrating factor

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In this talk we analyze the structure of the Poincaré map Π associated to a monodromic singularity of an analytic family of planar vector fields. We work under two assumptions. The first one is that the family possesses an inverse integrating factor that can be expanded in Laurent series centered at the singularity after a weighted polar blow-up fixed by the Newton diagram of the family. The second one is that we restrict our analysis to a subset of the monodromic parameter space that assures the non-existence of local curves with zero angular speed. The conclusions are that the asymptotic Dulac expansion of Π does not contain logarithmic terms, indeed it admits a formal power series expansion with a unique independent generalized Poincaré-Lyapunov quantity, which can be computed under some explicit conditions. Moreover we also give conditions that guarantee the analyticity of Π, in which case we show that the Bautin ideal is principal and therefore the cyclicity of the singularity with respect to perturbation within the family is zero.

Keywords: Monodromic singularity; planar vector field; inverse integrating factor; Poincaré map.

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Well-balanced POD reduced-order models for nonlinear systems of PDEs

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This work explores the application of Proper Orthogonal Decomposition Reduced Order Models (POD-ROMs) to hyperbolic systems of balance laws, with a particular focus on efficiently addressing nonlinear terms [4]. While POD-ROMs offer substantial computational savings compared to traditional full-order methods (FOMs) based on finite volume schemes, the inherent nonlinearities in these systems pose significant challenges for accuracy and stability in reduced-order modeling. Other recent works, such as [1], have also dealt with this issue.

To overcome these difficulties, we propose a novel approach that combines the Discrete Empirical Interpolation Method (DEIM) [3] with the well-established Proper Interval Decomposition (PID) framework [2]. Through a series of numerical experiments involving Burgers' equation with a nonlinear source term and the shallow water equations with bottom topography and Manninig friction, we demonstrate the effectiveness of this DEIM-enhanced strategy in accurately approximating nonlinear dynamics, while significantly reducing spurious oscillations near discontinuities.

Additionally, we provide a theoretical result on the well-balancedness of ROMs, proving that if the FOM is exactly well-balanced, the derived ROM preserves this property.

The proposed methodology is also extended to parametric studies, enabling the construction of predictive ROMs for hyperbolic systems dependent on physical parameters. In particular, we evaluate the model's performance in estimating Manning's friction coefficient.

Keywords:Reduced order models, Proper Orthogonal Decomposition, hyperbolic systems of balance laws, finite volume schemes.

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Numerical resolution of some stiff problem

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In this work we will present some numerical results corresponding to stiff problems of different nature. As it is well known, stiff problems require numerical methods where the choice of the discretization parameter is chosen for stability reasons and not for accuracy reasons. We will study three stiff problems that mathematically correspond to a hyperbolic partial differential equation problem, a slow-fast dynamic system and a slow-fast stochastic system. For the first case, we will combine methods for stiff problems such as TR-BDF2 or Newmark, for discretization in time, with finite elements for discretization in space. For the second, we developed the multirate technique for solving the problem, due to its slow fast nature. Finally, we will develop a stochastic version of the TR-BDF2 method that preserves the order of convergence for the solution of the system.

Keywords: Stiff problem; Slow-fast dynamic systems; Numerical analysis; Stochastic differential equation.

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Numerical study of wave reflection in the aortic artery

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Cardiovascular diseases are the main cause of death in the world, accounting for approximately 50% of total non-communicable diseases [1]. Hypertension and arterial blockage in its different forms are some of the most common ones. Behind these pathologies there are many physical phenomena, among which is the wave reflection.

Some studies point to three main factors that cause it: narrowing (for example, stenosis), branching and arterial wall stiffness [2]. The physical consequences that eventually lead to the aforementioned cardiovascular problems are increase of systolic pressure, decrease in the diastolic pressure and increase in perfusion pressure [3].

This work focuses on the study of these effects caused by wave reflection in the case of the aortic artery. The goal is to quantify the relevance of each of the three causes of reflection, in order to know which are the most decisive.

To approach the problem we start with a very simple model and add complexity until we get as close as possible to the real physiological characteristics of the aortic artery. The simpler situation is a tube with rigid walls [4], that is, without taking into account the elasticity of the artery wall. The problem is solved with a multi-scale computational approach [5], combining a 3D Computational Fluid Dynamics (CFD) model and a lumped-parameter model (Windkessel). In this simpler case, as it is an only-fluid problem, only the Navier-Stokes equations are solved:

$$div(\mathbf{u_f}) = 0 \quad \text{in } \Omega_f$$

$$\rho_f \frac{D\mathbf{u_f}}{Dt} + \boldsymbol{\nabla} p - div(T_\tau) = 0 \quad \text{in } \Omega_f$$

and completed with the 3-element Windkessel model for the pressure outlet condition:

$$Q\left(1 + \frac{Z_C}{R_p}\right) + CZ_C\frac{dQ}{dt} = C\frac{dP}{dt} + \frac{P}{R_p}$$

Once the fluid model is properly calibrated, a two-way fluid-structure interaction (FSI) problem can be modelled by adding the elastic arterial wall and its corresponding Cauchy's eqilibrium equation:

$$\rho_s \frac{\partial \mathbf{u_s}}{\partial t} - div(\sigma) = 0 \quad \text{in } \Omega_s$$

subject to Dirichlet conditions of non-displacement at the ends and with the correct coupling at the FSI interface imposing kinematic and dynamic boundary conditions:

$$\mathbf{u_f} = \mathbf{u_s} \quad \text{in } \Gamma_{FSI}$$

$$\sigma\hat{n} = T_\tau\hat{n} - p\hat{n} \quad \text{in } \Gamma_{FSI}$$

After calibrating the FSI model, different geometrical configurations and mechanical properties can be considered to assess the importance of each of the causes of wave reflection and their possible relationship with the most common cardiovascular diseases (see Fig. 5).

This approach will provide a deeper understanding of wave reflection phenomena in the aorta, which could open the door to new strategies for early prevention, diagnosis and treatment of different pathologies. Furthermore, the proposed methodology has the potential to be applied in other biomedical studies related to blood flow and FSI.

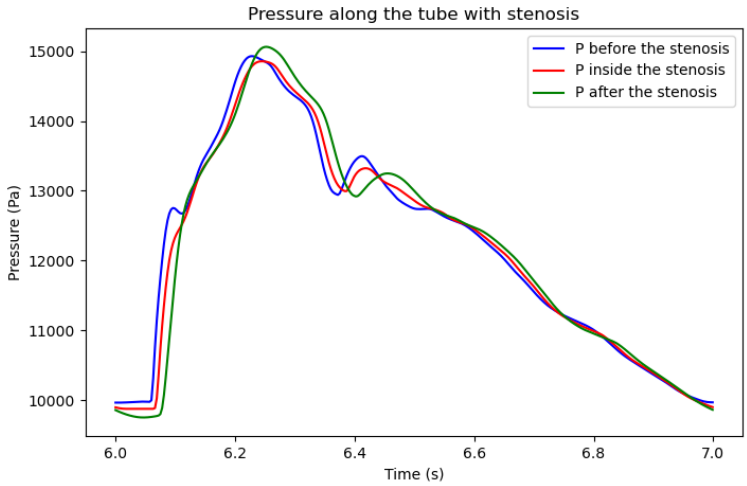

Figure 5. Pressure profiles at different cross sections of a stenotic tube. The reflected wave generated by the stenosis is added to the incident wave both upstream (blue) and downstream (green) of the stenosis, causing a pressure increase that does not occur in the stenotic section (red)..

Keywords: wave reflection; aortic artery; Windkessel; Computational Fluid Dynamics (CFD); Fluid-Structure Interaction (FSI)

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Bifurcations of symmetric $p:q$ resonant orbits

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We perturb two-dimensional conservative maps with symmetric elliptic orbits and study degenerate $p:q$ (with integer numbers $p < q$ and q odd) resonances. We consider different kinds of perturbations. We estabish that reversible and non-conservative perturbations lead to the appearance of non-symmetric dissipative periodic orbits. This is crucial for the emergence of reversible mixed dynamics, the new type of chaos characterized by the inseparability of dissipative and conservative elements of dynamics.

Keywords: bifurcations; conservative systems; reversible systems; elliptic orbits; resonance.

Using Lagrangian Coherent Structures to assess the impact of freshwater fronts in the Toconao pellet spill

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The use of Lagrangian models for the simulation of drifting objects in the ocean is very useful for simulating various situations that occur in the oceans (search and rescue, pollution events, larval and plankton dispersion, etc.). The Lagrangian particle dispersion model implemented in the open source state-of-the-art OpenDrift code (https://opendrift.github.io/) allows offline simulations to be carried out using available stored output of hydrodynamic and/or atmospheric model runs as forcing. In this way, a reanalysis of events of interest can be performed with the Lagrangian model.

A set of Lagrangian simulations were performed to study the dispersion and arrival of the sacks of pellets spilled from the Toconao container ship in December 2023. Due to the uncertainty about their buoyancy characteristics and their state (whether they were sacks or pellets drifting by their own), the sacks of pellets were simulated as particles without properties and as several Leeway drifting objects. We have checked that the trajectory model is influenced by the choice of the simulated particle type as well as the considered models used as forcing, which has already been established (see [1]). In addition, we have also concluded that our experiments only explain the spill (based on a compilation of available data), if particles are forced by both a hydrodynamic and an atmospheric models.

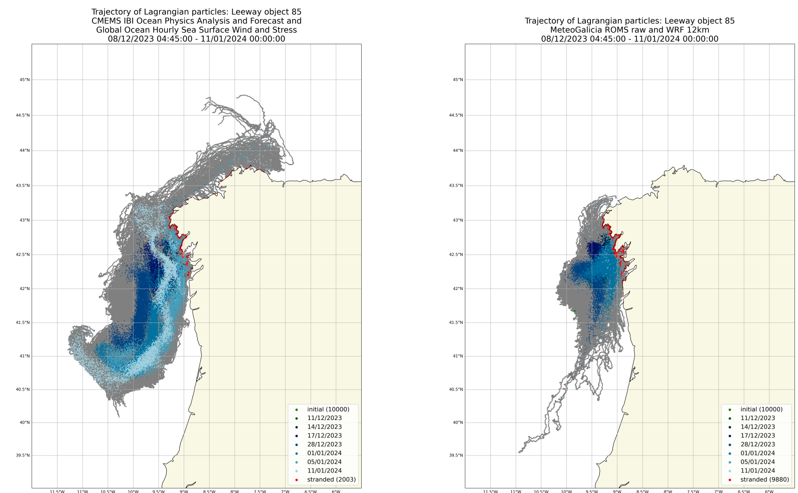

Figure 6. Simulation of pellets dispersion forced by CMEMS (left figure) and MeteoGalicia (right figure) hydrodynamical and atmospheric models..

As the Lagrangian simulations forced by two different hydrodynamic models yield different particle dispersion (see figure 6), we studied the differences between the two models: Atlantic-Iberian Biscay Irish- Ocean Physics Analysis and Forecast provided by Copernicus Marine Environment Monitoring Service (CMEMS) and ROMS raw provided by the regional meteorological agency MeteoGalicia.

In this contribution, we will present an analysis of the differences in these ocean models. Wind events and the presence of freshwater have a major impact on the description of currents in the models, especially in regions such as the Galician Rías (see [3]). Using numerical techniques to characterise Lagrangian barriers, we have analysed the outputs of the two considered models, trying to assess the impact of existing freshwater fronts at the time of the spill. Our results illustrate that the existence and evolution of Lagrangian Coherent Structures (LCS, see [2]) at the spill period of time are influenced by river plumes. The different LCSs and salinity fields modeled by both hydrodynamic models explain the different results of the Lagrangian simulations forced by both models (see figure 7). Consequently, our results show that LCSs are a tool for characterising freshwater fronts and studying the differences of dispersion and interaction with the coast of the simulated spill.

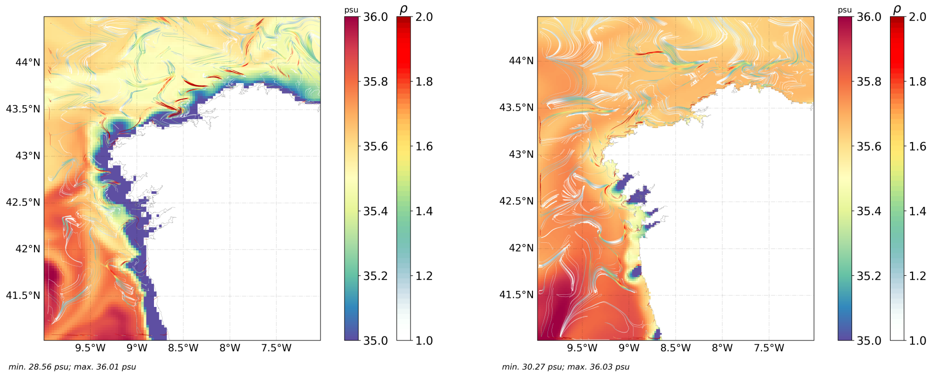

Figure 7. LCSs detected on CMEMS (left figure) and MeteoGalicia (right figure) hydrodynamic models with their respective salinity fields on the background at a given time..

Keywords: Lagrangian simulation; Ocean dynamics; Lagrangian Coherent Structures.

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Nonequilibrium criticality in the dynamics of synchronization in one dimension

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The study of synchronous dynamics has traditionally focused on the transition to synchronization from a static (steady-state) perspective, while the dynamical *process* whereby systems of oscillators synchronize at long times has been much less studied. One might expect this process to be strongly system-dependent, yet we have recently shown that it displays robust universal features previously observed in the context of nonequilibrium critical dynamics, as suggested by a mathematical connection between spatially discrete oscillator models and the continuum equations of surface kinetic roughening [1]. By means of detailed numerical studies of 1D systems of phase [1, 2, 3] and limit-cycle oscillators [4, 2] in the presence of quenched [1, 4] and time-dependent noise [2], or a combination of them [3], we provide evidence confirming that the synchronization process in these systems is characterized by forms of generic scale invariance associated with kinetically rough interfaces, such as that of the Kardar-Parisi-Zhang (KPZ) universality class [2]. In fact, the precise form of the coarse-grained dynamical equations and the role of symmetries and randomness can be analytically understood by a combination of continuum approximations and phase-reduction methods [4, 3]. We moreover find that fluctuations around the average growth generically follow a Tracy-Widom distribution, frequently associated with the KPZ nonlinearity. Synchronization and surface growth are more closely related than previously anticipated due to such robust universal features, which make the experimental observation of the nonequilibrium criticality of synchronization an alluring possibility.

Keywords: synchronization; Kardar-Parisi-Zhang universlity: nonequilibrium critical dynamics; surface kinetic roughening; generic scale invariance.

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Poisson-Lie groups and dynamics of nonlinear Hamiltonian dynamical systems

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A Poisson–Lie group (G, Π) is a Lie group G endowed with a Poisson structure Π such that the multiplication map $\mu : G \times G \to G$ is a Poisson map. The Poisson bivector Π induces in $\mathfrak{g} = \mathrm{Lie}\, G$ an adjoint 1-cocycle δ with values in $\wedge^2\mathfrak{g}$, and the pair $(\mathfrak{g}, \delta)$ is called a Lie bialgebra. The dual map to δ defines a Lie algebra structure on $\mathfrak{g}^*$ and thus, associated to any Poisson–Lie group (G, Π) there is a dual Poisson–Lie group (G^*, Π^*).

Given a Hamiltonian system $(\mathcal{H}, \mathcal{M}, \pi)$ defined of a Poisson manifold $(\mathcal{M}, \pi)$, the theory of Lie bialgebras and their associated Poisson-Lie groups can be used to construct infinite families of Hamiltonian systems that can be thought as formal deformations of the original one. The first step of the procedure essentially amounts to find a diffeomorphism $\varphi : \mathcal{M} \to \mathfrak{g}^*$ such that $\varphi_* \pi = \pi_{LP}$, where the Poisson brackets associated to π_{LP} are isomorphic to $\mathfrak{g}$. This Hamiltonian system $(\mathcal{H} \circ \varphi, \mathfrak{g}, \pi_{LP})$ is the one corresponding to an Abelian dual Poisson–Lie group, or equivalently to the trivial original Poisson–Lie group $(G, 0)$. The second step considers non–Abelian Poisson–Lie dual groups (G^*, Π^*) (equivalently, non-trivial original Poisson–Lie group (G, Π)), which generate smooth deformations of the original Hamiltonian system $(\mathcal{H}, \mathcal{M}, \pi)$. Remarkably, these deformed systems maintain the integrability of the original one, but are at first sight completely different.

This approach has been previously applied to some interesting examples [1, 2, 3]. Also, in [4, 5], the Poisson–Lie structure of (G^*, Π^*) is shown to be related with the existence of invariant volume forms, and therefore with the integrability, of all Hamiltonian systems defined on (G^*, Π^*), including those obtained from the procedure described above.

However, a detailed study of the nonlinear dynamics of these systems has not yet been performed. In this talk, which is based on ongoing work in collaboration with Angel Ballesteros and Alfonso Blasco, the first results of this study will be presented, highlighting the prominent role played by the center of the Poisson algebra of the dual Poisson–Lie group (G^*, Π^*) in relation with the qualitative behaviour of the dynamical system, as reflected on its phase space. To illustrate these results different deformations of epidemiological models (see [6, 7]) will be explicitly shown.

Keywords: Nonlinear dynamics; Systems of ordinary differential equations; Integrability; Poisson geometry and Casimir functions; Lie bialgebras; Poisson–Lie groups; Deformations of Hamiltonian systems; Phase space analysis.

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Regularity Results on Anisotropic PDEs

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In this presentation, I start by revisit several previous works concerning anisotropic differential equations, namely results towards solutions regularity when considering the PDEs

$$u_t - \operatorname{div}\left(u^{\gamma(x,t)} Du\right) = 0 \ , \quad \gamma(x,t) > 0 \ ;$$

$$u_t - \sum_{i=1}^{N} \left(u^{m_i}\right)_{x_i x_i} = 0 \ , \quad m_i > 0 \ ;$$

written for nonnegative bounded functions u defined in $\Omega_T = \Omega \times (0, T]$, being Ω a bounded domain in $\mathbf{R}^N$ and $0 < T < \infty$.

From these works a recent collaboration, with S. Ciani and I. Skrypnik, emerged. I'll briefly discuss the ongoing projects related to the study of these and other anisotropic PDEs, pointing out some of the difficulties one needs to address and overcome.

Keywords: Nonlinear PDEs; Anisotropic PDEs; Singular or degenerate PDEs

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The phenomenon of quenching in a system with non-local diffusion

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Diffusion models appear in multiple sciences such as biology, physics or even economics. They come up naturally as a broad class of natural processes, like the transport of matter due to random molecular motions. The most common diffusion operator in dynamical systems is the laplacian Δ, which is derived from Fick's laws and leads us to the local diffusion model $u_t = \Delta u$. It is called local diffusion because this model only considers the possibility of the particles moving very short distances in each instant of time. However, there can be more than that. There are other phenomena, such as the propagation of a pathogen, in which the particles could jump long distances in each instant of time thanks to various means of transport. We call this non-local diffusion and it is modeled by different operators, such as those of the type $J * u - u$, where the kernel J is a density function of the probability of any jump happening, see [1]. The most famous non-local operator is the fractional laplacian $(-\Delta)^s$, which has by itself spawned a whole array of literature such as [7].

The phenomenon of quenching in a dynamical system consists of the explosion of the velocity of the solution while the solution itself remains bounded. It was first assessed by Hideo Kawarada in 1974 for the equation $u_t = u_{xx} + (1-u)^{-1}$, where it happens whenever the solution reaches the value $u = 1$, see [5]. The phenomenon of quenching appears naturally in physical models such as the nonlinear heat conduction in solid hydrogen, see [2], or the Arrhenius Law in combustion theory, see [3]. It is for this reason that quenching has been the subject of numerous studies since Kawarada's paper, as seen in the surveys [2, 6].

The aim of this talk is to speak about our study of a system of equations with weakly coupled singular absorption terms and a non-local diffusion operator of the type convolution with a smooth kernel and the quenching phenomena that arises. First we will offer a suitable introduction to the non-local diffusion operator and the quenching phenomenon so that the talk can be followed by anyone interested but without prior knowledge on these topics. Then we will show our results about the system, which tackle the appearance of stationary solutions, the quenching rates of both components, the possibility of both components presenting quenching at the same time and the added difficulties this problem presents with respect to the single equation with non-local diffusion studied in [4].

Keywords: quenching; non-local; absorption; stationary; rates.

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Quiescent states of a granular fluid in an external potential

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It is well known that a classical fluid under the influence of steady and conservative external forces reaches an equilibrium state. It is a steady state, macroscopically characterized by having a space-dependent density and a spatially uniform temperature. Moreover, the velocity field is zero, i.e. it is a **quiescent** state. But, how is the situation when the fluid is out of equilibrium?

We try to answer the above question when considering a granular fluid [1], a system made up of macroscopic particles (grains) that dissipate energy through short-range collisions. Therefore, when subject to external energy injection, a granular fluid is always out of equilibrium.

We assume the granular fluid is well described by a system of hydrodynamic, Navier-Stokes equations. That is a system of nonlinear partial differential equations for the density of particles, the velocity field, and the granular temperature (a measure of the local mean kinetic energy of grains). The equations take into account the energy dissipation and injection, by considering generic functional dependence on the density and temperature, and on the external potential.

We find conditions under which the Navier-Stokes system admits steady and quiescent solutions. Moreover, we show that, when the external potential is not “too symmetrical”, a quiescent stationary state does not exist, rather than simply being unstable, and correspondingly, a steady convective state emerges spontaneously. The role of the walls/boundary conditions and the connection to self-diffusiophoresis are also addressed.

Keywords: Granular fluid; Hydrodynamics; Quiescent states; Nonlinear PDEs.

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Optimal control applied to viral competition

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The emergence of viral mutations presents a significant challenge in epidemiology and public health, as these variants often exhibit increased transmissibility and resistance to existing treatments [1, 2]. In this work, we propose a mathematical framework to model viral competition dynamics and introduce an optimal control strategy to mitigate the spread of mutant strains [3].

We consider a system of ordinary differential equations (ODEs) based on a modified Lotka-Volterra competition model [4],

$$\frac{dV_A}{dt} = r_A\left(1 - \frac{V_A + V_B}{k_A}\right)V_A - c_A u(t) V_A,$$
$$\frac{dV_B}{dt} = r_B\left(1 - \frac{V_A + V_B}{k_B}\right)V_B - c_B u(t) V_B,$$

where V_A and V_B represent the viral densities of the original and mutant strains, respectively. The parameters r_A, r_B denote reproduction rates, k_A, k_B are competition coefficients, and c_A, c_B measure the effectiveness of antiviral treatment $u(t)$, which serves as the control variable [5].

The goal is to minimize the impact of the mutant strain while reducing intervention costs, modeled by the cost functional,

$$J = \int_0^{t_f} \left[(V_A - k_A)^2 + (V_B - 0)^2 + u^2(t)\right] dt.$$

We apply Pontryagin's Maximum Principle [6] to derive the optimal control strategy and solve the system numerically using direct optimal control methods. Our results show that a properly designed control can significantly suppress the mutant strain while maintaining the overall viral population at manageable levels [7], as shown in Figure 8. These findings highlight the crucial role of biomathematical models in designing efficient intervention strategies against emerging viral threats.

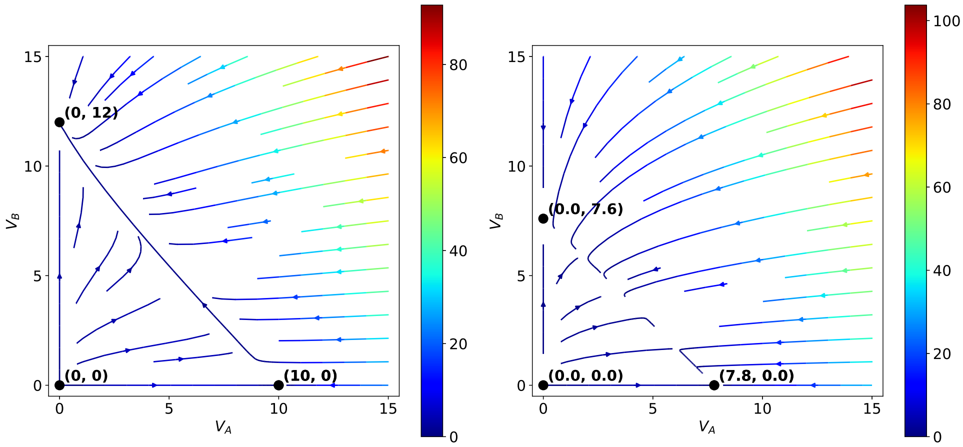

Figure 8. Phase portraits for the uncontrolled model (left) and the controlled model (right)..

Keywords: Mathematical epidemiology; viral competition; optimal control; differential equations; public health.

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Lyapunov vectors as optimal paths in random media

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The evolution of infinitesimal perturbations in spatially extended chaotic systems has been shown [1, 2, 3, 4, 5, 6] to be generically described by the heat equation with multiplicative noise. After a logarithmic (Hopf-Cole) transformation, the statistical description of the dynamics of perturbations is captured by the prototypical stochastic surface growth equation of Kardar-Parisi-Zhang [7]. In the surface picture, erratic fluctuations, due to the chaotic nature of the trajectory, are treated as an effective noise. In Ref. [3] it was shown that the correlations associated with covariant Lyapunov vectors (CLVs) (other than the first one) in extended systems with spatio-temporal chaos exhibit scaling with the wavenumber k as $\sim k^{-\delta}$ with exponent $\delta \approx 1.30$ at long wave-lengths ($k \to 0$). It was also observed that the crossover from KPZ scaling, $\delta = 2$, to the new universality with $\delta \approx 1.30$ takes place at shorter length scales as one looks at higher order CLVs [3]. This new scaling exponent has been shown [6] to be crucial to explain the universal scaling of Lyapunov-exponent fluctuations in space-time chaos. While the correspondence between the main LV and KPZ universality can be understood through the multiplicative heat equation ansatz, the origin of the asymptotic $\delta \approx 1.30$ scaling for sub-leading LVs has remained an open question for the last fifteen years.

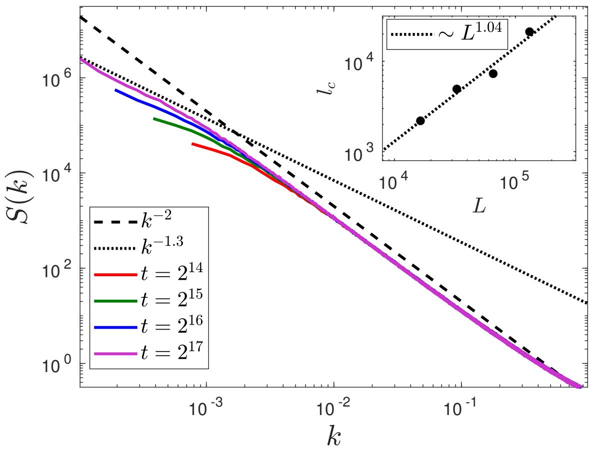

Figure 9. Power spectral density $S(k,t)$ of energy fluctuations for optimal directed paths of lengths $t = 2^{14}$ to 2^{17}. Data are averaged over 10^4 noise realizations. Dash and dotted lines are guides to the eye corresponding to the exponents $\delta = 2$ and $\delta = 1.3$, respectively. The inset shows how the estimated crossover length $l_c = 2\pi/k_c$ varies with the system size, $l_c \sim L^{1.04}$ is obtained from a linear fit..

Here we study the problem of directed paths in random media (DPRM) at zero temperature [8], which ground state is known to be in the same universality class as KPZ, after a suitable correspondence between the free energy of the minimal path and KPZ surface height [8]. We study, by means of numerical simulations, the excited states of the DPRM at $T = 0$, i.e. those paths with energies larger than the optimal path (ground state). We show [9] that the DPRM energy profile E(x), which includes the energies of all paths (i.e. including excited states), starting at (0,0) and ending at (x,t), exhibits fluctuations that scale as $\langle|\widehat{E}(k)|^2\rangle \sim k^{-\delta}$, where the exponent crosses over from $\delta = 2$ for large k to $\delta \approx 1.30$ at long wavelengths $k \to 0$. Our results [9] strongly support a link between the CLVs in space-time chaos and the excited states of the DPRM problem. We conjecture that free energies of the DPRM excited states map into surface heights of the CLVs.

Keywords: Spatio-temporal-chaos; Lyapunov vectors; Nonequilibrium statistical mechanics; Finite-size scaling

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Fourth order problem with functional perturbed clamped beam boundary conditions

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In this talk we will study the existence of positive solutions of the following nonlinear fourth order boundary value problem

$$u^{(4)}(t) + M\,u(t) = f(t, u(t), u'(t)), \quad t \in [0,1],$$

coupled to functional perturbed clamped beam boundary conditions:

$$u(0) = \lambda_1\,L_1(u), \quad u(1) = \lambda_2\,L_2(u), \quad u'(0) = \lambda_3\,L_3(u), \quad u'(1) = -\lambda_4\,L_4(u),$$

where M, λ_i are real parameters and L_i are linear operators, $i = 1, \dots, 4$. We will use two different approaches, with two different operators: the first method is based on the fixed point index theory, while the second one uses Schauder's fixed point theorem.

The results are published on the reference [1].

Keywords: nonlocal boundary conditions; fixed point index; Green's functions.

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New conserved quantities and modern symmetry analysis applied to a dissipative Westervelt's equation

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Propagation of sound waves in a compressible medium has several important applications where nonlinear and dissipative effects are relevant. Examples are parametric arrays in water and in air, under water imaging, musical acoustics of brass instruments, sonochemistry, quality control and characterization of materials, and bio-medical devices. Especially significant is ultra-sound imaging in human tissue. A simple mathematical 1D model is given by a dissipative version of Westervelt's equation describing the pressure fluctuation. Symmetries and conservation laws are intrinsic, fundamental aspects of wave equations. Their existence is not precluded by dissipative and nonlinear effects. The present work is devoted to illustrating some of these developments for the dissipative Westervelt equation:

- Lie point symmetries of the dissipative Westervelt equation,
- conservation laws of the dissipative Westervelt equation,
- construction of the potential system,
- potential Lie point symmetries,
- potential conservation laws,
- variational structure.

Keywords: conservation laws; conserved integrals: nonlinear acoustics; symmetries.

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Implicit Schwarz domain decomposition method with Legendre collocation for a Rayleigh-Bénard problem

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Schwarz domain decomposition (SDD) methods are widely used in numerical resolution of PDE systems [2, 3]. The original domain is split into several smaller subdomains that are solved separately. In this particular work, each PDE system is solved using a Legendre collocation method [1]. Among domain decomposition, alternating methods are the most widely used [3]. There, each subdomain is solved independently using the information previously known from adjacent subdomains. Nevertheless, as every subdomain changes, the result will include some errors. To improve the solution, this process is repeated, creating an iterative algorithm. However, large problems with multiple subdomains can cause the stagnation of the errors [2]. To overcome this inconvenient, an implicit Schwarz domain decomposition method is introduced here.

The implicit algorithm solves every subdomain at the same time allowing the perfect transmission of information. Therefore, the algorithm finds the exact global solution using only one iteration. Implicit SDD does not only overcome the stagnation of the errors, but completely eliminates them. Moreover, due to the particular structure of the linear system generated in the implicit SDD algorithm, a specific Gauss method can be applied to decrease the computational cost significantly. In conclusion, the implicit Schwarz domain decomposition method presented achieves error-free solutions in the same amount of time as its alternating counterpart.

In this work, implicit SDD has been used alongside a Legendre collocation method to solve the Rayleigh-Bénard convection problem in a rectangular domain Ω. The Oberbeck-Boussinesq approximation and infinity Prandlt number are considered, leading to the dimensionless equations,

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p - Ra\theta \mathbf{e}_z &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \\ \partial_t \theta + \mathbf{u} \cdot \nabla \theta - \Delta \theta &= 0, \end{aligned} \tag{2}$$

where $\mathbf{u}$ is the velocity field, p the pressure, θ the temperature and Ra the Rayleigh number.

The presented algorithm has been used to find a large variety of solutions for the Rayleigh-Bénard problem (2) with great success, including some studies on turbulent regimes. As an example, Figure 10 includes the temperature isotherms of an intermittent turbulent solution of (2) for $Ra = 4.5 \cdot 10^5$.

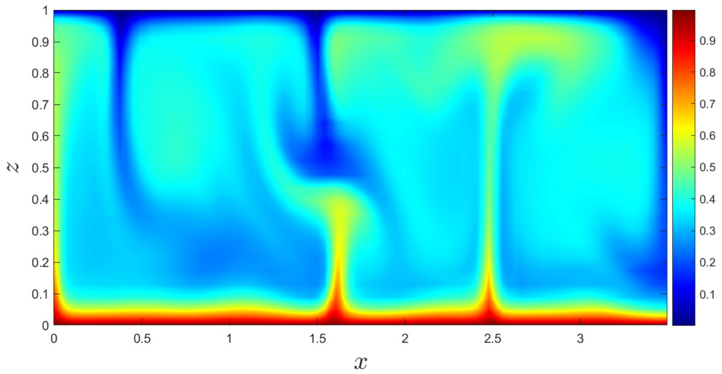

Figure 10. Isotherms of the Rayleigh-Bénard problem for $Ra = 4.5 \cdot 10^5$.

Keywords: Fluid Mechanics; Numerical methods; Domain decomposition; Spectral methods.

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Pure-Lagrangian methods for multiphysics simulation of electric upsetting processes

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This work presents a new approach for the numerical simulation of Electrically Assisted Forming (EAF) processes. The methodology is built upon a purely Lagrangian formulation and uses implicit Runge-Kutta schemes for time integration. A comprehensive model based on continuum mechanics is designed to couple thermal, electrical and mechanical interactions while accommodating large deformations. To achieve this, a fully coupled thermo-electro-mechanical Lagrangian framework is introduced, which includes an elasto-viscoplastic constitutive model and considers several independent unknowns: velocity, temperature, electric potential, the plastic deformation gradient, an internal variable representing strain hardening, and a Lagrange multiplier enforcing contact constraints. The choice of a pure-Lagrangian formulation brings notable advantages, such as a known computational domain that remains fixed, the simplified tracking of free surfaces and the elimination of convective terms. For spatial discretization, finite element techniques are employed. The proposed numerical method is validated through various two-dimensional simulations involving cylindrical symmetry, where convergence rates in both time and space are analysed using benchmark problems. Furthermore, for an in-die electric upsetting process, the computed results closely align with those obtained from commercial simulation tools, demonstrating the effectiveness of the approach.

Keywords: thermo-electrical-mechanical; large deformations; time dependent domain; pure-Lagrange-Galerkin methods; electric upsetting.

Acknowledgements: The research has been developed in collaboration with CIE Galfor through a project granted by CDTI and signed between the company CIE Galfor and Itmati (nowadays, integrated in CITMAga). This work has been partially funded by Xunta de Galicia under research Project GI-1563 ED431C 2021/5, and MCIN /AEI /10.13039/501100011033 / FEDER, UE under research Project PID2021-122625OB-I00.

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Non-smooth saddle-node bifurcations for a family of piecewise-linear and quasiperiodically forced maps

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In this work we present a class of two-parameter family of quasiperiodically forced one-dimensional maps $F_{a,b}(x,\theta) = (f_{a,b}(x,\theta), \theta + \omega \mod 2\pi)$, where x is real, $\theta \in \mathbb{T}$ is an angle, and $\omega \in \mathbb{R} \setminus 2\pi\mathbb{Q}$ is an irrational frequency, and that $f_{a,b}(x,\theta) = h_a(x) - b(1 + \cos(\theta))$ is a real piecewise linear map with respect to x where h_a is given by

$$h_a(x) = \begin{cases} -\pi/2 & \text{if } x \in (-\infty, -\pi/2a), \\ ax & \text{if } x \in [-\pi/2a, \pi/2a], \\ \pi/2 & \text{if } x \in (\pi/2a, \infty). \end{cases} \tag{3}$$

This work is related to the recent paper [3] where it is studied a similar family of quasiperiodically forced maps, with the difference in the perturbation function. This family is a "simplified" version of a family for which $f_{a,b}(x,\theta) = \arctan(ax) - b(1 + \cos(\theta))$. In the case of this family when a is large and b is increased, two invariant curves approach and they start to wrinkle until they seem (numerically) to become a strange set when they merge for a critical value. In the works [1, 2, 4], the authors examine various scenarios in which the invariant curves exhibit similar behavior. In some cases, the existence of a Strange Non-Chaotic Attractor (SNA) can be rigorously proven, while in others, only numerical evidence of its presence can be obtained. In our case, the family $F_{a,b}$, with h_a as in (3), depends on two real parameters, $a > 0$ and $b \in \mathbb{R}$. The significance of this work lies in the fact that, for this family, we establish both the existence and uniqueness for $b > 0$ of a critical value $b = b^*(a)$ at which a non-smooth saddle-node bifurcation occurs. For $a < 1$ and any b there is only a Lipschitz invariant curve. For $a > 1$ there exists a smooth map $b = b^*(a)$ such that: a) For $b < b^*(a)$, $f_{a,b}$ has two Lipschitz attracting invariant curves and one analytic repelling one; b) For $b = b^*(a)$ it has one analytic repelling invariant curve, one Lipschitz attracting invariant curve and one semicontinuous (non-continuous) attracting invariant curves, the analytic repelling curve and the attracting semicontinuous (non-continuous) curve intersect in a zero-Lebesgue measure and residual set of angles; c) For $b > b^*(a)$ it has one Lipschitz attracting invariant curve. The case $a = 1$ is a degenerate case that is also discussed in the work. It is important to highlight that we give an explicit expression for the value $b^*(a)$ and, in case (b), it can be rigorously proven that there exists a family of Lipschitz invariant curves indexed by the parameter b that exhibit a fractalization process as the parameter b approaches $b^*(a)$ from below; see [5] for further details about fractalization of curves. In the present work, since our map is only Lipschitz continuous, we analyze the fractalization phenomenon in terms of the unbounded growth of the Lipschitz constant of the attracting invariant curve as $b \to b^*(a)^-$. This perspective is particularly significant, as numerical computations clearly indicate that the Lipschitz constant of the attracting invariant curve increases as b approaches $b^*(a)$ from below. This observation provides strong evidence supporting the fractalization process of invariant curves as a method for the creation of Strange Non-chaotic Attractors.

Keywords: Strange nonchaotic attractor; nonsmooth bifurcation; pitchfork bifurcation; quasi- periodic forcing; piecewise-linear.

Acknowledgements: This work has been supported by the Spanish grant PID2021-125535NB-I00 funded by MICIU/AEI/ 10.13039/501100011033 (FEDER/UE), and J.C.T. have also been supported by the Catalan grant 2021 SGR 01072.

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Computation and stability analysis of periodic orbits using finite differences, Fourier or Chebyshev spectral expansions in time

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Identification of periodic orbits (POs) of nonlinear dynamical systems, together with the computation of their stability, is crucial in various domains of physics and engineering. Many numerical approaches to the determination of periodic orbits in ordinary differential equations (ODEs) exist. Popular choices include shooting methods and their variants see Chap 10 in [4]. We are interested here by another class of methods that expresses the periodic orbit (PO) in a "global" manner. Among them, one of the most widely used alternative methods is a Fourier-based Galerkin method known as Harmonic Balance Method (HBM) e.g. [2]. HBM approximates the periodic orbit as a truncated Fourier series. Due to the spectral expression of the cycle, high accuracy or equivalently fast convergence can be achieved. Historically, HBM was used mainly for low-dimensional ODE systems in mechanical and electrical engineering. Recent studies have considered such methods for higher-dimensional systems stemming from the spatial discretisation of partial differential equations (PDEs), in particular from fluid mechanics [1, 6, 3]. The linear stability of an HBM solution is usually analyzed by forming the so-called Hill matrix and analyzing its eigenvalues in the search for unstable Floquet exponents. As pointed out by several authors see e.g. [5], a major drawback of using Hill's matrix is that only a subset of the computed eigenvalues are relevant. As a consequence, it is necessary to filter out the spurious exponents from the relevant ones. Various strategies have been proposed in low-dimensional [5].

We analyse and compare several algorithms to compute numerically periodic solutions of high-dimensional dynamical systems and investigate their Floquet stability without building the monodromy matrix. The solution and its perturbation are discretised in time either using finite differences, Fourier-Galerkin or Chebyshev expansions. The resulting nonlinear set of equations describing the periodic orbit is solved using a Newton-Raphson algorithm. The linearised equations determining the stability lead to a generalised eigenvalue problem. Unlike the Fourier-Galerkin method, the use of Chebyshev polynomials or finite differences has the advantage that the relevant Floquet exponents are directly given without the well known issue of having to sort out the eigenvalues. The speed of convergence of these three methods is illustrated with examples from the Lorenz system, the Langford system and a two-dimensional thermal convection flow inside a differentially heated cavity. This last example demonstrates the potential of the newly proposed Chebyshev expansion for large-scale problems arising from the discretisation of the incompressible Navier-Stokes equations.

Illustration for Lorenz system: The Lorenz system is governed by a system of three ordinary differential equations:

$$\frac{dx}{dt} = s(y - x), \quad \frac{dy}{dt} = x(\rho - z) - y, \quad \frac{dz}{dt} = xy - \beta z.$$

Here we fix $s = 10$, $\beta = 8/3$ and vary the value of ρ. For two values of ρ, we illustrate in figure 11 how the proposed Chebyshev method is able to compute a (stable or unstable) periodic orbit, its period T and the associated Floquet exponents and eigenvectors.

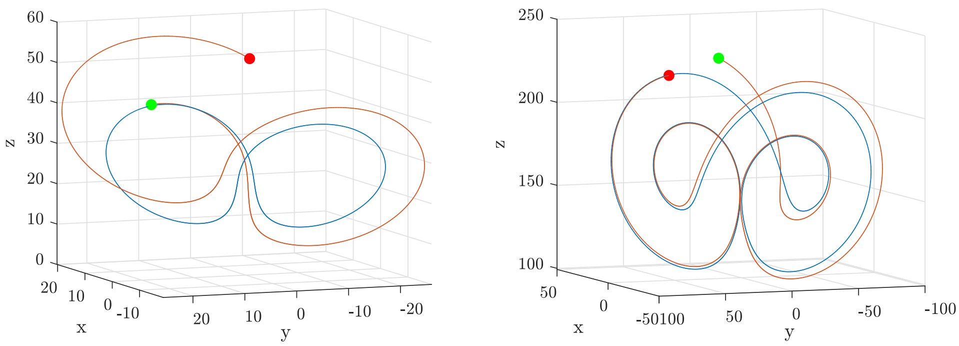

Figure 11. Periodic orbit solution of the Lorenz system for $\rho = 28$ ($T = 1.559$)(left) and $\rho = 160$ ($T = 1.153$)(right) in blue, together with an unstable (left panel) or stable (right panel) eigenvector (solid red line) computed with the Chebyshev method. Green dot: initial position, red dot: position after one period.

Keywords: Periodic orbits; Dynamical sytems; Stability analysis; Floquet analysis; Chebyshev polynomials; Harmonic Balance; Hill's method; Navier-Stokes equations.

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Plowing induced ripple formation in compliant materials

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In 1992 Leung and Goh [1] published a pioneering work in which they showed the ability of an AFM tip to generate a series of wavy patterns, which have been known as‘ripples", when traversing certain polymeric surfaces. In these systems, the effect of the AFM on the surface is to mold the surface itself, which deforms under the pressure exerted by the tip. Leung and Goh's work opened an important experimental line in the field, which has led to remarkable advances. However, a theoretical interpretation of the basic underlying mechanisms that allows characterizing the obtained patterns and understanding and studying their origin has been elusive for many years. We will present here a summary of our contribution to the field. In particular we will report on our first simulations based on a 3d simple model and its comparison to experimental results [2]. We will abound on numerical studies on the formation and evolution of ripples and other patterns [3]. And we will focus on our theoretical and experimental work on the influence of geometry and study of border effects in the building and formation of the ripples [4]. Finally we will present recent results on the characterization of ripple formation in rough surfaces and our current numerical approach to the understanding of the experimentally observed nucleation and detachment of polystyrene particles in our samples [5].

Keywords: pattern formation; nonlinear dynamics; material science.

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Stability of nonlinear Dirac solitons under the action of external potentials

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The nonlinear Dirac equation in 1+1-dimensions supports localized solitons. Theoretically, these traveling waves propagate with constant velocity, energy, momentum, and charge. However, the soliton profiles can be distorted, and eventually destroyed, due to intrinsic or numerical instabilities. The constants of motion and the initial profiles can also be modified by external potentials, which may give rise to instabilities.

In this work [1], we study the instabilities observed in numerical simulations of the Gross-Neveu equation [2] under linear and harmonic potentials. We perform an algorithm [3] based on the method of characteristics to numerically obtain the two soliton spinor components. All studied solitons are numerically stable, except the low-frequency solitons oscillating in the harmonic potential over long periods of time. These instabilities are identified by the non-conservation of both energy and charge, and can be removed by imposing absorbing boundary conditions. We find that the dynamics of the soliton is in perfect agreement with the prediction obtained using an Ansatz with only two collective coordinates. By applying the same methodology, we also demonstrate the spurious character of the reported instabilities in the Alexeeva–Barashenkov–Saxena (ABS) model [4] under external potentials.

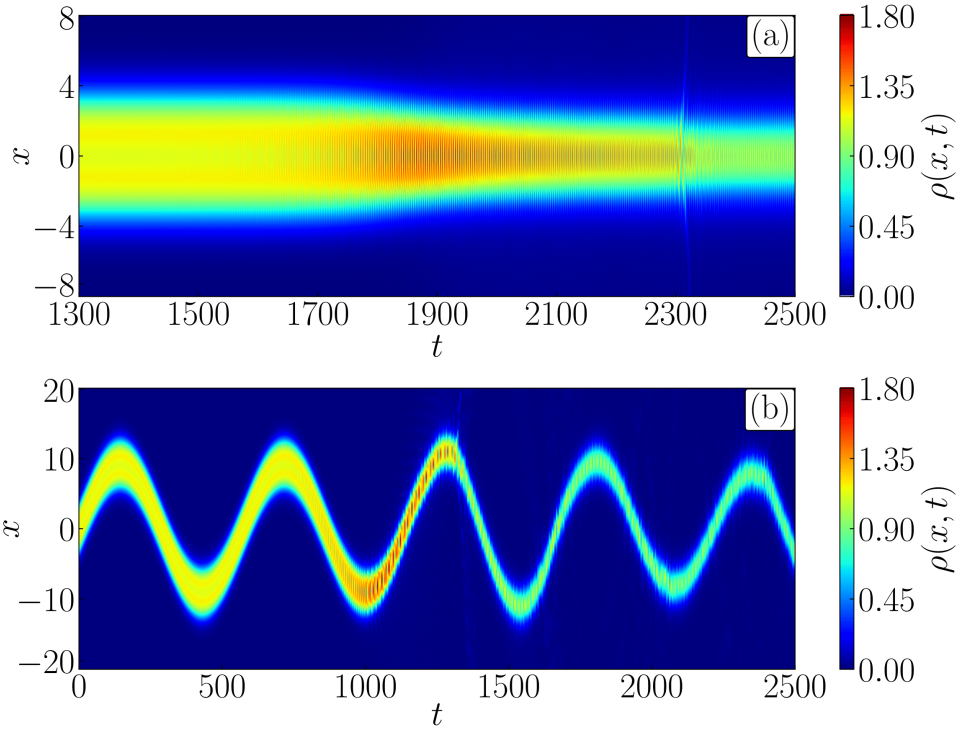

Figure 12. Evolution of the density of the charge from simulations of the ABS model (a) without and (b) with harmonic potential..

Keywords: non-linearity; Dirac equation; solitary waves; collective coordinate theory; numerical methods; stability.

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Moist convective vortices: intensification by condensation

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This paper numerically examines the formation of convective vortices in a cylindrical domain with a localized cooled region at the top and studied the effect of moisture condensation on the morphology and intensity of the vortex generated. The model used in this work is rotating Rayleigh-Bénard convection, widely used for studying the fundamental processes underlying atmospheric phenomena such as tornadoes. Following the model in [1] we include the effect of condensation of moist air into this model. We demonstrate that the transition from a one-cell vortex structure to a two-cell vortex observed in the dry air configuration when thermal conditions are varied, is accelerated when condensation occurs. In addition, the vortex velocities intensify through the condensation process, which is explained via a force balance analysis. The fundamental results presented contribute to the understanding of intensification mechanisms in some atmospheric vortices.

Keywords: vortex; thermal convection: condensation; vortex intensification.

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Exploring dynamics in FitzHugh-Nagumo systems

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In the context of a linear coupling of FitzHugh-Nagumo systems:

$$\left\{\begin{array}{rcl} x_1' & = & c\,(y_1 + x_1 - \frac{1}{3}x_1^3) + \alpha_1(x_2 - x_1), \\ y_1' & = & -\frac{1}{c}(x_1 - a + b\,y_1) + \alpha_2(y_2 - y_1), \\ x_2' & = & c(y_2 + x_2 - \frac{1}{3}x_2^3) + (\alpha_1 + \varepsilon_1)(x_1 - x_2), \\ y_2' & = & -\frac{1}{c}(x_2 - a + b\,y_2) + (\alpha_2 + \varepsilon_2)(y_1 - y_2), \end{array}\right.$$

the unfolding of codimension-two Hopf-Hopf singularities is discussed in order to show the presence of a wide range of different non-resonant cases established by theory [1]. Among others, two in which the bifurcation diagram includes invariant 2 and 3-tori are present. Exploration of dynamics beyond the singularities reveals an extraordinary richness of behaviors, including quasiperiodic dynamics and resonance phenomena. Continuing the Neimark-Sacker bifurcations linked to the Hopf-Hopf singularities, chaotic regions are detected, including boundary crisis bifurcations where sudden size changes occur in the attractor, as well as regions where the chaotic attractors exhibit synchronization phases.

Keywords: Hopf-Hopf bifurcations; Neimark-Sacker bifurcations; border collisions; synchronization.

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Thermoregulated organ-on-a-chip via magnetic hyperthermia

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Spatial and temporal temperature variations drive numerous phenomena and cell behaviours, especially when dealing with chemical reactions or cells. Finding mechanisms to study and reproduce these temperature changes in organs-on-a-chip (OoC) opens the door to new experiments and setups. Although temperature control is traditionally achieved by submerging the OoC in a constant-temperature bath, we demonstrate an alternative approach: embedding magnetic nanoparticles directly into the polydimethylsiloxane (PDMS) matrix to serve as integrated heat sources. However, the impact of this integration on heat production and its transfer to a flowing fluid within the OoC remains unclear.

To find out, we designed an OoC with the aforementioned characteristics and reproduced and calibrated its digital twin with experimental data, which we then used to estimate the mechanisms of heat delivery of the thermoregulated OoC to the flowing liquid in its channel via Computer Fluid Dynamics (CFD). Our *in silico* studies reveal that the low thermal conductivity of PDMS significantly influences temperature distribution at a cross-section of the channel, making a temperature gradient appear at the interface of the liquid with the OoC when flowing through half a channel sealed with a slide, an effect difficult to achieve with a conventional thermal bath but potentially helpful in studying processes such as cell migration.

Our findings demonstrate that integrating heat generation within the chip is feasible, enabling new experimental designs. For example, combining pure and nanoparticle-doped PDMS regions within a single chip could allow simultaneous control and test conditions within the same experiment, expanding the versatility of OoC platforms.

This contribution is based on our recent findings presented in [1], pending publication.

Keywords: Thermoregulable; Organ on a Chip; Microfluidics; CFD; Hyperthermia; Magnetic Nanoparticles.

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Modeling Microbial Interactions with Generalized Lotka-Volterra Equations: Identification Challenges

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In nature, microorganisms do not exist in isolation: they constantly interact with one another, forming intricate networks. Mathematical models provide powerful tools to study these dynamics and extract meaningful insights about microbial populations. Among the various modeling approaches, *Lotka-Volterra* equations have been widely used to study microbial communities, as well as in other fields such as economics (to model competition between firms), ecology (to describe predator-prey dynamics), epidemiology (to study disease spread in interacting populations), and neuroscience (to model neural competition and synchronization). *Generalized Lotka-Volterra* (*GLV*) models extend the classical predator-prey framework, providing a more flexible representation of species interactions. These models consist of a system of nonlinear, coupled, first-order differential equations that capture both interspecies interactions and external influences. Mathematically, the model is expressed as:

$$\frac{dX_i}{dt} = \mu_i \cdot X_i + X_i \cdot \sum_{j=1}^{n} \beta_{ij} X_j \quad (i = 1, ..., n)$$

where X_i represents the relative abundance of species i, μ_i its intrinsic growth rate, and β_{ij} quantifies the pairwise interactions between species i and j. Here we consider its application in quantitiative microbiology. In particular we consider case studies of varying levels of complexity, ranging from simple cases, such as a system of two influenza virus strains (*GLV2*) [1] or a synthetic three species community (*GLV3*) [2], to more complex ecosystems, such as the human gut microbiome (*MGLV*) [3].

We focus on the parametric identification of generalized Lotka-Volterra (GLV) models using time-series data, i.e. the parameter estimation problem. This process is typically framed as a nonlinear dynamic optimization problem, where the goal is to minimize the discrepancy between experimental data and model predictions. However, this task can be quite challenging due to several factors, including issues of identifiability (both structural and practical), the instability of the dynamics, the presence of local optima, and the risk of overfitting.

We give special attention to instability problems caused by finite-time blow-up, with an analysis of how this affects both the optimization convergence and the stability of the solutions. This blow-up phenomenon refers to the numerical instability where the model's solution diverges to infinity in a finite amount of time, leading to unrealistic or nonsensical results. Blow-up can be triggered by a variety of factors, such as incorrect initial conditions, inappropriate parameter values, or insufficient regularization of the model. One of the key challenges with blow-up is that it can happen even when the model appears to behave normally over short time scales, only to exhibit instability in the long run. An example can be seen in Figure 13, where we show an overfitted solution of the *MGLV* case study which also exhibits blow-up at the end of the time horizon considered.

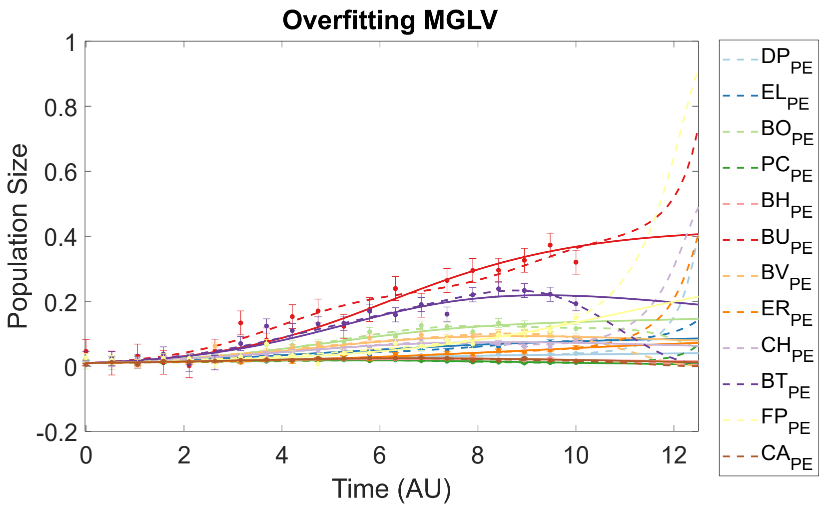

Figure 13. Case study *MGLV*: An example of overfitting (dashed lines) is shown in comparison to the fit using nominal parameter values (solid line). While the nominal model remains stable throughout, the overfitted model demonstrates blow-up behavior toward the end of the time horizon.

Finally, we introduce an identification pipeline that accounts for these challenges, offering a robust approach to overcome them. The effectiveness of this pipeline is demonstrated through the case studies discussed earlier. By integrating robust optimization techniques and addressing key stability concerns, we contribute to a more accurate and interpretable modeling framework. Future research may extend this methodology to incorporate additional constraints, leverage machine learning techniques for improved parameter inference, and explore broader applications beyond microbiology.

Keywords: Generalized Lotka-Volterra; Microbial Communities; Systems Biology.

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Computer-assisted proofs for subharmonic Melnikov functions with applications to the Earth-Moon-particle system

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Resonant periodic orbits are often used in astrodynamics and space mission design due to their predictable and repeatable behavior. For example, the nominal orbit of Lunar Gateway–the next-generation space station–is a 9:2 near-rectilinear halo orbit (NRHO) around Earth-Moon L_2, i.e., the station orbits the Moon 9 times in the time it takes the Sun to orbit the Earth-Moon frame 2 times. In the Earth-Moon system, the dynamical model used for preliminary mission design is the circular restricted 3-body problem (CR3BP)–an autonomous Hamiltonian system; however, there are significant perturbing effects, from the ellipticity of the Earth-Moon orbit to the gravitational influence of the Sun, that must be incorporated to ensure successful operation of the mission. Each of these effects can be modeled as periodic perturbations to the CR3BP–the Elliptic R3BP (ER3BP) or R4BP, respectively. In either case, most CR3BP periodic orbits persist as invariant 2-tori under certain non-degeneracy conditions prescribed by KAM theory; yet, resonant periodic orbits persist as resonant periodic orbits, often multiple orbits of the same resonance, as determined by subharmonic Melnikov theory. So, if there are multiple orbits with the same resonance in the perturbed (and more realistic) models, then which orbit is "the" 9:2 NRHO for Gateway? What is the relationship between the resonance of an orbit and the number of resonant orbits persisting into the perturbed system? Moreover, as these perturbations are not so small, are there bifurcations of these orbits, leading to a qualitative change in the dynamics?

In this work, we seek to prove the existence of a certain number of L_2 9:2 NRHOs in the Earth-Moon ER3BP. To do this, we break the problem down into several steps. First, we study lower-order resonant orbits in the same family–4:1 and 5:2. For each of the three resonant cases, we prove the existence of the periodic orbit in the CR3BP (the unperturbed problem) using computer-assisted techniques, e.g., the radii polynomial approach. Next, we show that the subharmonic Melnikov function, $\mathcal{M}$, of the ER3BP can be written in a particular form, enabling us to state and prove some pen-and-paper results for $\mathcal{M}$. The analytical results give insight into the relationship between the order of an orbit's resonance and the number of persisting periodic orbits (for small values of the perturbation), as well as a simplification of the evaluation of $\mathcal{M}$. As classical subharmonic Melnikov theory states rigorous results for lower-dimensional problems, we strive to state a subharmonic Melnikov theorem that has a classical flavor but for higher-dimensions; in other words, we develop a theorem relating the size of the neighborhood around the zeros of $\mathcal{M}$ with the size of perturbation magnitude and accuracy of prediction by $\mathcal{M}$. Then, applying the subharmonic Melnikov analysis to the three resonant orbit cases (4:1, 5:2, 9:2), we use numerical continuation to increase the perturbation magnitude to the "real" Earth-Moon ER3BP, wherein we re-work the radii polynomial approach to prove the existence of a particular number of resonant periodic orbits in the Earth-Moon ER3BP. This is a work-in-progress talk, and each of these steps may serve as a talk on their own. So, we work through only some of the details of the radii polynomial approach and subharmonic Melnikov theory, with the hope of communicating the flavor of our approach to solving an interesting problem in celestial mechanics and computational dynamics originating in a real-world engineering scenario (i.e., from astrodynamics, space mission design, and NASA).

Keywords: astrodynamics; computer-assisted proofs in dynamics; celestial mechanics; dynamical systems.

Advanced numerical methods for DED AM Simulation: multi-scale domain decomposition and moving meshes

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Additive manufacturing (AM) is a rising technology in the manufacturing industry. AM approach to manufacturing pieces is to built them adding material instead of the classical approach of substract it. This opens new possibilities in the industry, helping to reduce costs and wastage in the manufacturing processes of some pieces. Direct energy deposition (DED) is a kind of metal AM based on the simultaneous application of material addition and heat source. Its main advantages are the higher deposition rates and the freedom in dimensions of the pieces to be built. As a drawback, having a high power heat source continuously melting large portions of metal bring some challenges, for example, deformations and defects may appear due to the effect of the high thermal gradients on the metallic piece.

DED AM is a promising technology; however, it is not yet fully prepared for completely automated manufacturing processes. There is no state of the art tool that can effectively predict the outcome of a given manufacturing process. Such tool could enhance the industrial viability of this technology, expanding its use to new manufacturing sectors. The main approach for building pieces is to use the knowledge of professionals experienced in the process, along a series of trial and error sequences, to obtain a suitable strategy for manufacturing a given piece.

Simulation of DED AM processes faces several challenges that increase its computational costs, making them unmanageable. The deposited material is melted, forming a melt pool, where fluid dynamics and phase changes have a crucial role. As the heat source moves, the molten material solidifies, forming the desired piece. During the process, the entire piece is heated, causing high thermal gradients near the melt pool. The thermal evolution induces mechanical deformations and residual stresses, which accumulate throughout the fabrication.

Another factor that increases the computational costs of the DED AM simulations is the multi-scale behaviour of the process. Critical phenomena occur in a relatively small region compared with the size of the entire piece. This critical region, located just under the heat source, moves along with it throughout the building process. Additionally, as it is being built, the piece is continuously growing, what requires the computational domain to also grow accordingly. In this work, the proposed method is designed to address these two challenges, aiming to reduce computational cost. Only thermal evolution, with convection and radiation boundary conditions, will be taken into account in this first approach.

The selected strategy is to use the Arlequin method, a domain decomposition method, to divide the treatment of the problem into the critical region and the non-critical region. This allows us to use two separate meshes for each subdomain: a finer, moving mesh to capture the critical behaviours and a coarser mesh to handle the rest of the piece.

To further leverage this domain decomposition, we introduce a change to a reference frame on the moving Arlequin subdomain, which implicitly accounts for the domain growth. Additionally, we use the Finite Adition of Matter Element method (FAME), introduced in [1], to update the coarse mesh at discrete time steps. This approach enables the use of a quasi-static mesh for the coarser subdomain, reducing computational costs.

The methodology will be validated through a series of analytical tests designed to replicate the key behaviors observed in real applications.

Keywords: Domain Decomposition; Multi-scale Methods; Additive Manufacturing; Arlequin Methods; Finite Element Method.

Acknowledgements: This work was partially supported by FEDER and Agencia Española de Investigación funds under the PID2023-146841OB-I00 grant and by GAIN and Xunta de Galicia funds under the IN606D grant. The authors would also like to acknowledge AIMEN technology centre for their help to understand the complex industrial process that was modeled in this work.

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A study of the Taylor coefficients of stiff ODEs

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In this talk we study the behavior of the coefficients of the Taylor method for the solution of an ODE system

$$y' = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, T]$$

that is *stiff* in the sense of Söderlind, Jay and Calvo [3]. We use the computational technique called *jet transport* [2] to study how the use of a floating-point arithmetic induces an error in the Taylor coefficients related to the variational equations of the ODE. This error can exhibit dominant exponential growth in stiff problems, and it is naturally related to the stiffness indicator introduced by Söderlind *et al.* This behavior is illustrated by numerical experiments on classical stiff problems. In particular, we study the implications of this effect when high-order Taylor methods are used for extended precision computations.

Keywords: stiffness indicator; taylor method; jet transport; extended precision.

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An alternating Schwarz domain decomposition method applied to the Rayleigh-Bénard convection problem

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In this work we present an alternating Schwarz-type domain decomposition (DD) method applied to the Rayleigh-Bénard convection problem in a 2D fluid layer. The problem is modeled with the incompressible Navier-Stokes equations together with the heat equation in a rectangular domain and the Boussinesq approximation is considered [6, 7]. The problem is solved numerically with a Legendre collocation method and the mesh used is defined by Legendre-Gauss-Lobatto collocation points [1, 2]. Since the spectral methods are ill-conditioned it is necessary to use a DD method by dividing the problem into smaller subdomains and, in this way, the ill-conditioning of Legendre collocation is overcome. Then, solutions can be obtained for large aspect ratios and/or for larger Rayleigh numbers [3, 5] in order to achieve the turbulence. The stationary and the evolution problem using the Schwarz alternating DD method and the solutions with this method are compared with the reference solutions obtained with finite elements by COMSOL Multyphysics [4] and turbulent solutions are shown.

Keywords: turbulence; Rayleigh-Bénard convection; Legendre collocation; domain decomposition method; alternating Schwarz.

Acknowledgements: This work was partially supported by the Research Grants PID2023-152004NB-I00 (Spanish Government) and 2021-GRIN-30985 (Universidad de Castilla-La Mancha), which include RDEF funds. D. Mart´ınez has a predoctoral contract from the Universidad de Castilla-La Mancha, which includes ESF+ funds.

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Modelling telomeres in stem cell populations

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Since tracking human stem cells is complicated, we studied their evolution through an in-silico approach, in order to obtain information that is useful in the laboratory. Models based on differential equations to simulate the generational and temporal evolution of stem cell populations were proposed in [1]. The maximum proliferation potential, the rate of mitosis, death events and telomerase activity were the parameters of the discrete mathematical model based on ordinary differential equations and the associated continuous model formed by a partial differential equation. The mean proliferation potential at each time was suggested as an indicator of population ageing. Numerical experiments studied the generational-temporal evolution of the mean proliferation potential in healthy individuals with different telomere length percentiles and different levels of telomerase activity. Then, the evolution in patients with primary ovarian insufficiency, with low telomerase activity, following treatment to reactivate either endogenous or exogenous telomerase were simulated.

Keywords: aging; telomere length; telomerase activity ; stem cells; primary ovarian insufficiency.

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On existence and localization of solutions for nonlinear systems

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In this talk, we consider the existence of solutions for operator systems of the form

$$\begin{cases} u_1 = T_1(u_1, u_2), \\ u_2 = T_2(u_1, u_2), \end{cases}$$

where $T = (T_1, T_2) : A \times B \subset X \times Y \to X \times Y$ is a compact map and X, Y are normed spaces. We deal with a hybrid localization of solutions based on a fixed point theorem for operators defined in Cartesian products which combines Krasnosel'skiĭ type compression-expansion conditions with Schauder fixed point theorem, see [1]. Its proof relies in an adequate application of the Leray-Schauder fixed point index in retracts. In this way, it is possible to obtain a solution $(\bar{u}_1, \bar{u}_2)$ such that the components $\bar{u}_1$ and $\bar{u}_2$ are localized independently.

As an application, we provide a novel localization of solutions for a system of second-order equations subject to homogeneous Dirichlet boundary conditions. It combines, in a component-wise manner, the method of lower and upper solutions with the localization provided by compression–expansion type fixed point theorems in cones. The details can be found in [2].

Keywords: Nonlinear systems; Fixed point index; Krasnosel'skiĭ's theorem; Schauder's theorem; Dirichlet problems.

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A Lagrangian IBM model for modelling Early Life Stages of different fish and cephalopod species

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In recent years, the use of Early Life Stages (ELS) models of fish species, which simulate eggs and larvae as Lagrangian particles evolving in an environment described by an oceanographic model coupled to an ecological one, has become widespread in oceanography. This biophysical model accounts for horizontal and vertical advection and dispersion of particles, and also for biological behavior from a Individual Based Model (IBM) approach. Biological processes such as changes in buoyancy throughout the development of the egg stages, vertical larval migration, larval growth as a function of larval stage, with effects of temperature and food availability, larval migration for feeding... are simulated. These models have become tools to analyse the variability of oceanographic conditions and their effect on plankton as well as on egg and larval survival. Therefore, these models allow the study of recruitment and connectivity of fish stocks, which are issues of great scientific and fisheries management interest.

Particle models for larval population dispersal studies are highly dependent on the target species, although from a numerical point of view if a model is available for a species with a pelagic phase, the work required to get a model for other species is just to adapt the numerical code to the life history characteristics of the other species. In this contribution we will show how we have used this approach to simulate early life stages of different fish species: the Iberian sardine (*Sardina pilchardus* [2], the European anchovy (*Engraulis encrasicolus*) and the European hake (*Merluccius merluccius*) and also of the cephalopod common octopus (*Octopus vulgaris*). During this ELS phase, eggs and larvae develop and are dispersed in an environment described by a hydrodynamical and ecological 3D model (see examples in 14).

In this contribution, we will review how in the framework of the Spanish project DEMON, together with ICM-CSIC colleagues, we have revised the available information on egg and larval stages for sardine and anchovy and we have reviewed different mathematical functions to simulate the different processes involved in the early life stages of these two species of small pelagic fish. In projects PHYS2FISH and CLONES, financed by the collaborative Galician Marine Sciences programme, the early life stages of the fish species hake and of the cephalopod common octopus are modeled in collaboration with fisheries specialists. For fish species, a model of early life stages complies an egg stage and a larval stage in which larvae evolve from yolk-sac larvae to self-feeding autonomous larvae, and we have revised how to model these stages and the survival of larvae. In the case of the cephalopod common octopus, the egg phase is benthonic and we concentrate in the modelling of a pelagic paralarvae phase after egg hatching.

As a result of these effort, a common code to simulate the ELS of the these four commercial species: sardine, anchovy and hake (fish) and common octopus (a cephalopod) is available as a module in the open source particle tracking model OpenDrift [3] that can be forced by the output of available hydrodynamical simulations. In this respect, we will illustrate how differences in hydrodynamical model configuration (initiation, resolution, forcing, and simulation domain) result in differences in the transport of Lagrangian particles [1] and how this fact impacts the distribution of fish larvae with the consequence that differences in hydrodynamical models need to be properly evaluated.

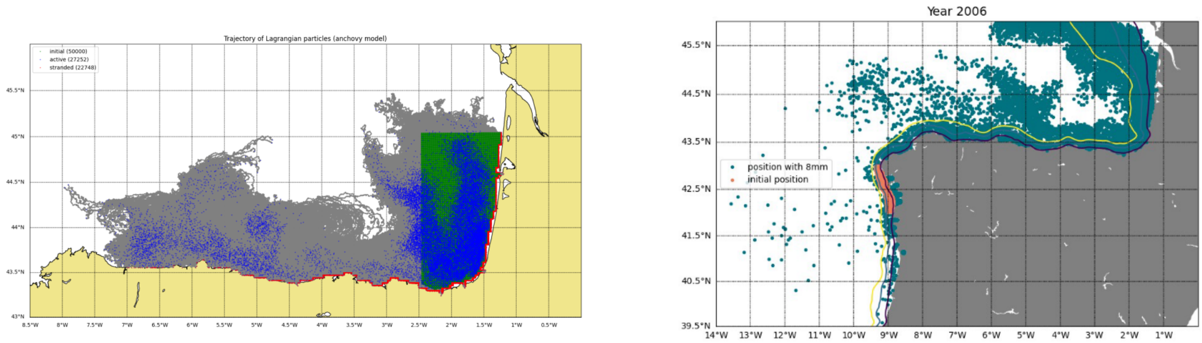

Figure 14. Dispersion of anchovy eggs and larvae in the Bay of Biscay (left) and of octopus paralarvae in Galicia (right).

Keywords: Biophysical ocean models, Individual Based Models, Fish Early Life Stages, Lagrangian particle dispersion

Acknowledgments: This contribution is part of project DEMON (Dissipation of Energy in Ocean Models and Connectivity) funded by MCIN/AEI/10.13039/501100011033 and by ERDF. Additional funding has been obtained from Phys2Fish project of the Galicia Marine Science programme of the Complementary Science Plans for Marine Science of Ministerio de Ciencia, Innovación y Universidades included in the Recovery, Transformation and Resilience Plan (PRTR-C17.I1). Funded through Xunta de Galicia with NextGenerationEU and the European Maritime Fisheries and Aquaculture Funds.

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Exploring spatiotemporal traffic dynamics in 25 cities via transfer entropy–driven causal networks

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The rapid evolution of modern urban societies has led to a significant rise in traffic congestion, emphasizing the need for more intelligent and adaptive transportation systems. It is now evident that simply expanding road infrastructure is insufficient to solve congestion challenges. To understand the complex and dynamic behavior of urban traffic, a multidisciplinary approach —integrating data analytics, physics, and social science— is essential.

In this contribution, I will build upon our recent study [1], where we analyzed traffic flows in 25 metropolitan areas across Europe, North America, and East Asia using the *UTD19 dataset*. This dataset compiles daily time series from over 10,150 Loop Detectors, resulting in more than 147 million traffic flow measurements. Through clustering techniques and statistical analysis, we identified two primary traffic regimes—morning inbound and afternoon outbound—and uncovered regional and cultural similarities in traffic patterns across cities.

Now, we take a step further and we will present our ongoing work aimed at constructing causality networks derived from the concept of transfer entropy. This approach allows us to infer directional interactions between traffic flows time series data, allowing us to perform a spatiotemporal analysis of the system. We employ a robust statistical framework, including null models based on block bootstrap techniques to simulate surrogates of the data, to ensure the validity of detected causal links (see Figure 15).

This causality-based perspective not only enriches our understanding of urban traffic dynamics but also opens new avenues for proactive traffic management and forecasting. By identifying key nodes that drive congestion patterns, our approach can support more targeted and effective interventions for urban planners and policymakers.

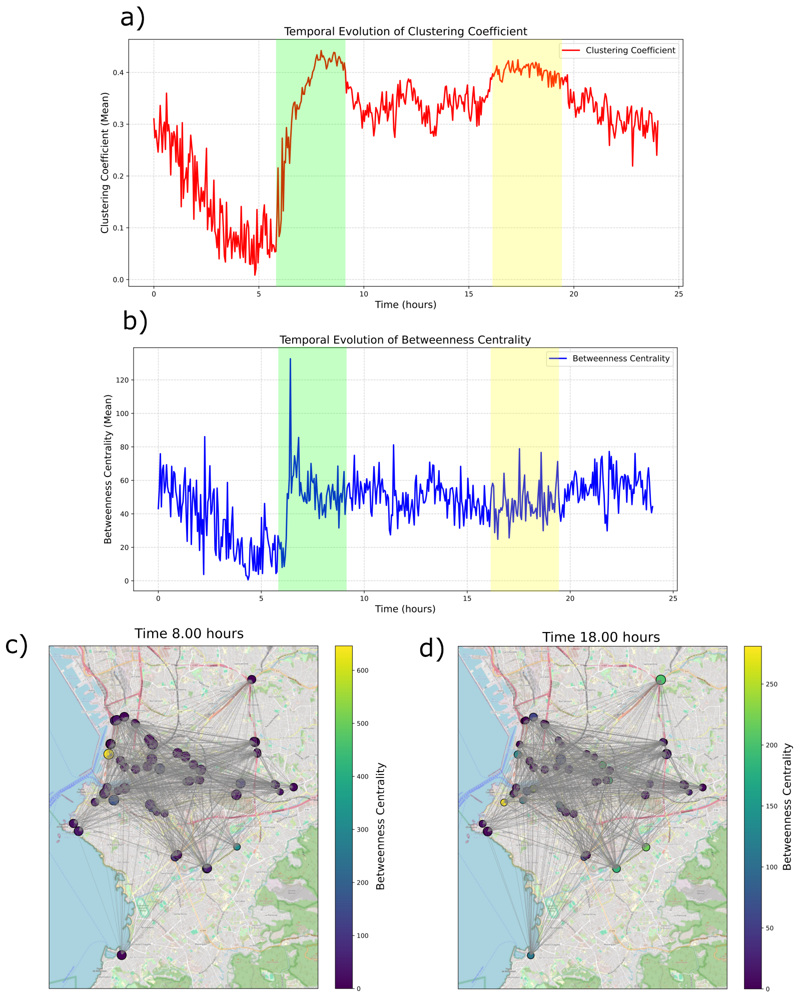

Figure 15. The figure shows the causality network for the morning (c) and afternoon (b) traffic flow rush hours. The nodes represent traffic sensors (loop detectors), while the edges indicate significant causal links between them. The color of the nodes represent the betweenness centrality. This can tell us which are the critical links at certain times of the day for the city traffic flow. The size of the nodes is proportional to their degree, representing the number of connections they have in the network. (a) and (b) are the average clustering coefficient and betweenness centrality of the network. Green and yellow shading represent the morning and afternoon traffic flow peak hours, respectively.

Keywords: Traffic congestion, Causality networks, Transfer entropy, Urban dynamics, Data-driven analysis

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Chaotic motions to L_4 in the restricted planar circular three-body problem beyond the Routh mass ratio

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This talk deals with the Restricted Planar Circular 3-Body Problem (RPC3BP) close to the Lagrangian critical point L_4. The RPC3BP models the motion of a massless body under the gravitational influence of two massive bodies (the primaries having masses $\mu > 0$ and $1 - \mu > 0$, respectively) performing circular orbits, assuming that the massless body moves in the same plane as the primaries. In a rotating framework that fixed the position of the primaries (a rotating framework with the same period of the primaries, also called synodic coordinates), the Hamiltonian of the RPC3BP is an autonomous two-degree of freedom Hamiltonian having five critical points L_1, L_2, L_3, L_4, and L_5 called the Lagrangian points. Before this symplectic reduction, these critical points correspond to periodic dynamics associated with the massless body with the same frequency as the primaries.

Let $\mu_1 = \frac{1}{2}\left(1 - \frac{1}{9}\sqrt{69}\right)$ be the Routh critical mass ratio. It is well known that if $0 < \mu < \mu_1$ the point L_4 is a center-type critical point whereas for $\mu > \mu_1$ it is a complex saddle. This phenomenon is also called Hopf Bifurcation.

We are interested in values of the mass ratio $\mu > \mu_1$, where chaotic phenomena are expected to occur. To investigate this, we studied the two-dimensional stable and unstable manifolds associated with L_4, derived an asymptotic formula for the distance between these manifolds, and used it to prove the existence of Smale horseshoes close to L_4.

Keywords: splitting of separatrices phenomena; celestial mechanics; 3 body problem; chaotic motions.

Analysis and simulations of a time-delayed model for tumor-immune system interactions

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A mathematical model for tumor-immune system interactions is presented and analyzed. The model incorporates the effects of adoptive cellular immunotherapy and includes a delay term that accounts for the time lag in the adaptive immune response following tumor recognition [1, 2]. The inclusion of the delay renders the model less idealized, while introducing additional complexities into the dynamics and asymptotic stability of the system. Moreover, we consider logistic-type proliferative terms that regulate to prevent potential blow-up in the long-term behaviour of the cellular concentrations. We investigate the model from both analytical and numerical perspectives. In particular, we prove global well-posedness of solutions, determine equilibrium states, and analyse both local and global stability. We show that the delay can destabilize the system and examine the resulting stability switches.

Finally, numerical simulations of the delayed model are presented to explore the effect of the delay on the dynamics and stability, and to validate the analytical results.

Keywords: biomathematics; cancer progression; delayed equations; stability analysis

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Symmetry breaking and oscillations in simple models for polymerisation

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Some fundamental building blocks of living matter, such as amino acids and sugars show the preference of one of the two chiral states of an otherwise chemically identical molecule. The origin of this so-called biological homochirality is still a matter of debate, even after decades of intense theoretical and experimental work. At the heart of many theoretical approaches lies a mechanism responsible for spontaneous mirror symmetry breaking, leading finally to stable stationary states of given chirality. In this talk, we discuss several simple mathematical models for polymerisation processes. In the first one, designed to create symmetry breaking in a similar way as in the Frank model, we observe and discuss oscillations of the chiral states. We compare this to the phenomenon of chiral excursions, found in similar systems, but which do not show permanent symmetry breaking. In the second model, we show how the interaction of mechanical forces with nucleation-fragmentation processes can lead to symmetry breaking. In the third model, we observe transient oscillations that are compared to a specific experimental system.

Keywords: Nonlinear oscillations; symmetry breaking; polymerization reactions; biological homochirality.

Acknowledgements: The Authors acknowledge financial support received from Plan Nacional project PID2020-116846GB-C22

Integrating urban CFD wind simulations with meteorological predictions

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Urban environments are characterized by complex aerodynamic challenges due to the intricate interaction between buildings, orography, vegetation, and atmospheric conditions. Understanding airflow patterns within cities is vital for a wide range of applications, which range from urban design optimization to drone path planning. Computational Fluid Dynamics (CFD) simulations have emerged as a powerful tool to simulate airflow in urban areas, offering non-invasive insights into the generated airflow patterns.

In this work, we focus on the automated reconstruction of real urban environments and its application to CFD simulations. The results are confronted against ground-truth data obtained from meteorological stations, enabling a thorough assessment of the accuracy and reliability of the proposed CFD reconstruction model, which could have significant implications in a wide variety of applications.

The proposed methodology in [1] reconstructs urban geometries by integrating LiDAR and cadastral datasets. Initially, ground height is extracted from the LiDAR data. Then, building shapes from the cadastral dataset are extruded to a height determined by the average LiDAR point height within each structure. The presence of vegetation and water masses is modeled using

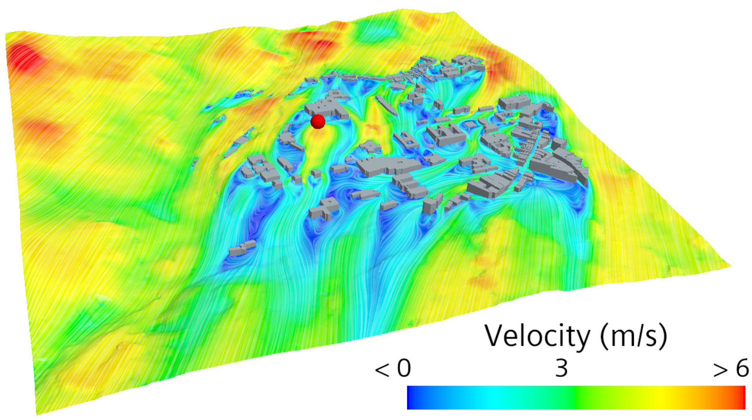

Figure 16. Simulation in the Campus Sur of the University of Santiago de Compostela. The red dot indicates the position of the meteorological station used for the validation process..

the roughness wall parameter from [2], while the boundary conditions are calculated following the Planetary Boundary Layer (PBL) equations [3].

An example of a CFD simulation can be seen in Figure 16. Once all data is collected, the geometry is automatically integrated into the CFD software, allowing wind field extraction at any point. To validate the results, data from a real meteorological station from the official Galician meteorological service (MeteoGalicia) is extracted. As seen in Figure 17, the findings indicate that OpenMeteo offers the most reliable simulations, as its continuously updated predictions ensure the use of the best available data. In contrast, MeteoGalicia and AEMET provide fixed forecasts that are not updated, making them true future predictions. While these services exhibit slight velocity overestimations, they maintain overall accuracy. Validation with meteorological station data confirms the model's effectiveness in capturing real-world wind patterns.

This study highlights the potential of integrating CFD and meteorological data for urban airflow analysis, offering valuable insights for city planners and environmental engineers. The proposed methodology presents a wide range of possible applications, including optimal path finding for drones, sound dispersion analysis in urban environments, fire spread modeling where wind plays a crucial role, optimization of ventilation strategies in indoor spaces or other areas where precise airflow modeling is essential.

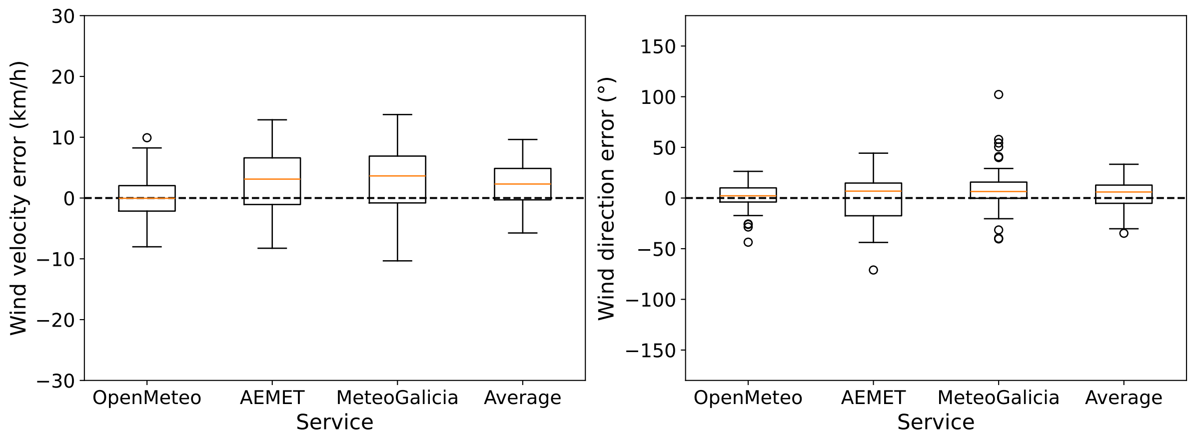

Figure 17. Boxplot of wind speed and direction errors for each meteorological service on a daily simulation over two months..

Keywords: Computational Fluid Dynamics; Urban physics; Terrain reconstruction.

Acknowledgements: The Authors acknowledge financial support received by the Doutoramento Industrial grant, from Xunta de Galicia (IN606D).

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A stabilized numerical method for the Darcy-Forchheimer model

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The Darcy-Forchheimer model is used to simulate fluid flow through porous media. It can be viewed as an extension of the Darcy model, which assumes a linear relationship between the fluid flow velocity and the pressure gradient. Nevertheless, this relationship is no longer linear for moderate to high velocities. The Darcy-Forchheimer model aims to account for this fact and incorporates a nonlinear term that becomes relevant at high fluid velocities. As a result, better predictions than the Darcy model are expected in those cases. Examples of industrial applications where the Darcy-Forchheimer model becomes useful include predicting petroleum flow in reservoirs, water movement in aquifers, and water migration during the drying of solids.

Let Ω be a bounded connected domain with a Lipschitz boundary Γ. We assume that Γ is divided into two disjoint parts, Γ_u and Γ_p. The problem reads: find the fluid velocity $\mathbf{u}$ and the fluid pressure p such that

$$\left\{ \begin{array}{rcll} \mu\,\mathcal{K}^{-1}\mathbf{u} + \beta\,\rho\,|\mathbf{u}|\mathbf{u} + \nabla p & = & \rho\,\mathbf{g}\,, & \text{in } \Omega\,, \\ \nabla\cdot\mathbf{u} & = & 0\,, & \text{in } \Omega\,, \\ \mathbf{u}\cdot\mathbf{n} & = & 0\,, & \text{on } \Gamma_u\,, \\ p & = & p_D\,, & \text{on } \Gamma_p\,, \end{array} \right. \tag{4}$$

where μ is the dynamic viscosity, $\mathcal{K}$ is the permeability tensor, β is the Forchheimer parameter, ρ is the fluid density, and $\mathbf{g}$ is the gravity force. We denote by $\mathbf{n}$ the unit outward normal to Ω and p_D denotes the prescribed pressure on Γ_p. Existence and uniqueness of a solution for the problem (4) was first proved by Fabrie [1]. Two main variational formulations are found in the literature: primal and dual-mixed. The primal-mixed formulation was introduced and analysed from theoretical and numerical point of view by Girault and Wheeler [2]; Salas et al. [4] proposed the use of different discrete spaces. The dual-mixed formulation was studied by Pan and Rui [3].

We propose a new stabilised dual-mixed finite element method for the numerical solution of the Darcy-Forchheimer model. The starting point is the dual-mixed formulation by Pan and Rui [3], which we stabilise by a penalty on pressure jumps across adjoining finite elements. The new discrete scheme allows to approximate the velocity by continuous piecewise linear polynomials and the pressure by piecewise constants. We proved the existence and uniqueness of a solution to the discrete scheme and a stability result. Besides, we derived optimal a priori error estimates. Finally, we tested the method through some numerical examples.

Keywords: Darcy-Forchheimer; finite element method; mixed formulation; stabilisation.

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Existence of solutions of nonlinear systems coupled to linear non-local boundary conditions

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In this talk we will study certain type of systems of differential equations subject to non local linear boundary conditions. In such boundary conditions, the dependence of certain parameters is considered. The idea of the study is to transform the given system into another first-order differential linear system together with two-point boundary value conditions. Then, we will obtain the explicit expression of the Green's function of the considered linear system with non-local boundary conditions. To do this, it is assumed that the Green's function of the homogeneous problem (that is, when all the considered parameters are zero) exists and is unique. In such a case, the homogeneous problem has a unique solution that is characterized by the corresponding Green's function g. We will show how the Green's function of the nonhomogeneous system is obtained as the sum of the function g plus another term that depends on the parameters involved in the boundary conditions and the expression of function g. As a consequence, the existence of solutions of nonlinear systems with linear non local boundary conditions is studied.

The novelty of our work is that the variables of the system do not appear separated neither in the equations nor in the boundary conditions. We will illustrate the obtained results with examples.

Keywords: Green's function; Nonlinear systems; Non-local boundary conditions.

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Posters

Heterodimensional cycles in three-dimensional diffeomorphisms

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We investigate generic unfoldings of the nilpotent singularity in $\mathbb{R}^3$, and demonstrate the existence of generic T-periodic perturbations such that the T-map defined by the flow at time $t = T$ has a dissipative heterodimensional cycle. We further discuss how such heterodimensional cycles relates to the emergence of strange attractors by adapting certain results given in [1].

Keywords: nilpotent singularity; heterodimensional cycle; strange attractor.

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Applications of physics-informed neural networks for solving differential problems: a comparisson with traditional methods

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In this work, we present applications of Physics-Informed Neural Networks (PINNs) for solving differential problems based on the approach given in [1]. Unlike traditional numerical methods, PINNs incorporate differential equations into the loss function, making them a versatile and increasingly popular tool in various fields of science and engineering. Said versatility can be used for solving both linear and nonlinear differential equations and systems, calculus of variations problems, control problems, and inverse problems, among others. For numerical experiments, the software tools Matlab, FreeFem, and Python, along with its TensorFlow module, are employed. Furthermore, we compare the results obtained with PINNs and those produced by classical methods.

Keywords: neural networks; differential equations; numerical analysis; PINNs.

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New insights into the global dynamics and the integrability of the Hide-Skeldon-Acheson system

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We present a work which focuses on the study of the Hide, Skeldon and Acheson dynamical system, which was proposed by Hide, Skeldon and Acheson in 1996 [3], and models a self-excited dynamo.

In [1], we provide by first time its invariant algebraic surfaces, first integrals and Darboux invariants.

To show the importance and usefulness of these results, we will illustrate how they allow us to study the global dynamics with some examples, one with a first integral and another with a Darboux invariant. To obtain the global phase portraits we use different techniques as the Poincaré compactification or the blow up desingularization [2].

Keywords: HSA system; invariant; first integral; global dynamics.

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Station-keeping on solar sail equilibria for a planetary alignment scenario

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In this work, we present a mission concept resistant to periods of solar conjunction with the aim of establishing communications between Earth and Mars. It can be accomplished by using two CubeSat nanosatellites equipped with a solar sail, a large orientable surface made of a lightweight and highly reflective material. Given the negligible gravitational attraction between both planets, we study the dynamics of the satellite in the separate Earth-Sun and Mars-Sun circular Restricted Three-Body Problem (RTBP) including the solar radiation pressure effect exerted on the sail. The equations of motion that describe the behavior of the considered CubeSat satellites are [2, 3]:

$$\begin{cases} \ddot{x} - 2\dot{y} = \dfrac{\partial \bar{V}}{\partial x} + \beta \dfrac{1-\mu}{r_1^2} \cos^2\alpha \left(-\dfrac{(x-\mu)z}{sr_1} \sin\alpha\cos\delta + \dfrac{y}{s}\sin\alpha\sin\delta \right), \\ \ddot{y} + 2\dot{x} = \dfrac{\partial \bar{V}}{\partial y} + \beta \dfrac{1-\mu}{r_1^2} \cos^2\alpha \left(-\dfrac{yz}{sr_1} \sin\alpha\cos\delta - \dfrac{x-\mu}{s}\sin\alpha\sin\delta \right), \\ \ddot{z} = \dfrac{\partial \bar{V}}{\partial z} + \beta \dfrac{1-\mu}{r_1^2} \cos^2\alpha \left(\dfrac{s}{r_1} \sin\alpha\cos\delta \right), \end{cases}$$

where α and δ are the orientation angles of the solar sail, β is the sail lightness number, μ the mass parameter of the RTBP, $s = (x-\mu)^2 + y^2$ and $\bar{V}$ the effective potential of the system, given by:

$$\bar{V}(x,y,z) = \frac{1}{2}(x^2+y^2) + (1-\beta\cos^3\alpha)\frac{(1-\mu)}{r_1} + \frac{\mu}{r_2},$$

with $r_1^2 = (x-\mu)^2 + y^2 + z^2$ and $r_2^2 = (x+1-\mu)^2 + y^2 + z^2$.

Due to the inherent non-linearities of the model (), we numerically determine the location of the admissible equilibrium points that are non-eclipsed in this Earth-Sun-Mars alignment by means of an iterative continuation method. These are the points where two CubeSat nanosatellites equipped with a solar sail could be placed. Since these equilibrium points are unstable, we consider station-keeping maneuvers in order to control the saddle behaviour and avoid the satellites from drifting away from the nominal position of the points. This control strategy relies on iteratively rectifying the sail orientation once a certain threshold distance from the equilibrium point is surpassed. To do so, the dynamics of the system is linearized around the unstable manifold of the equilibrium point to study the behavior of the system when the sail orientation is changed. This allows us to determine the best sequence of orientation changes by a least squares approximation performed on each step, as described in [1]. Therefore, the satellites are always kept close enough to the non-eclipsed equilibrium points, allowing to maintain active communications between the planets.

Keywords: Station-keeping; solar sail; equilibrium points; Restricted Three Body Problem (RTBP); solar radiation pressure; Earth-Mars communications; control strategy.

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Periodic Solutions for Nonlinear Systems via a vectorial version of Krasnosel'skiĭ's theorem

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In this work, we present a novel extension of Krasnosel'skiĭ's compression-expansion fixed-point theorem to product spaces. Motivated by the fixed point theorem in [1], concerning a vector version of the Krasnosel'skiĭ-Benjamin theorem, we establish sufficient conditions for the existence of a fixed point of a compact operator $T = (T_1, T_2)$ acting on a normed product space $X \times Y$, formulated explicitly in terms of norms in a componentwise manner. Our approach is based on a suitable computation of the Leray-Schauder fixed point index in cones and, to some extent, follows the line of [2].

A key feature of this generalization is that it allows the combination of compressive and expansive conditions separately on each component, overcoming a limitation of the classical Krasnosel'skiĭ theorem. In addition, our main result ensures the existence of a fixed point $(x_1, x_2) \in X \times Y$ such that $\|x_j\| > 0$ for each $j \in \{1, 2\}$.

As an application, we employ this result to establish the existence of T-periodic solutions with non-trivial components to a system of second-order differential equations of the form

$$\begin{cases} x'' + a(t)x = f_1(t, x, y), \\ y'' + b(t)y = f_2(t, x, y). \end{cases}$$

where $a, b \in L^p(0, T)$, with $1 \leq p \leq +\infty$, and the nonlinear functions $f_1, f_2 : [0, T] \times \mathbf{R}^2 \to \mathbf{R}$ are continuous. In this way, the existence criteria for scalar periodic second-order equations presented in [3] are adapted to the case of systems.

This is part of a joint work with Jorge Rodríguez-López (Universidade de Santiago de Compostela, Spain).

Keywords: Fixed point theory; Krasnosel'skiĭ's theorem; Non-trivial solutions; Periodic solutions.

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Nonlinear Bayesian model for early ovarian cancer detection

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Ovarian cancer is the most lethal of all gynecological cancers. When detected at an early stage, the survival rate is significantly higher than when diagnosed at an advanced stage. In this work, we provide evidence of enhanced performance using a combination of longitudinal biomarkers, compared to the best individual longitudinal biomarker, Cancer Antigen 125. To achieve this, we employ a nonlinear hierarchical Bayesian model for change-point detection across multiple biomarkers. Our data comprised 180 controls and 44 cases with serum samples (Cancer Antigen 125, Human Epididymis protein 4 and glycodelin) sourced from the multimodal arm of UK Collaborative Trial of Ovarian Cancer Screening (UKCTOCS).

Biomarker levels are represented by Y_{ijk}, where subject-specific variables are indexed by $i = 1, 2, ..., n_0, n_0 + 1, ..., N$, where n_0 represents the number of controls, while the remaining subjects are classified as cases. A specific biomarker is indexed by $k = 1, 2, ...K$. Each patient i has a screening visits t_{ij} where $j = 1, 2, ...T_i$.

For control patients, biomarker levels are expected to fluctuate randomly around a constant mean θ_{ik}, which is expressed as $Y_{ijk} = \theta_{ik} + \epsilon_{ijk}$, with $\epsilon_{ijk} \approx N(0, \sigma_k^2)$. For cases, we define a binary indicator I_{ik} to distinguish between two different model assumptions for biomarker evolution. If $I_{ik} = 0$, we assume that the marker level does not increase after cancer onset, following the same behavior modeled for controls. Conversely, if $I_{ik} = 1$, marker levels fluctuate around a mean θ_{ik} until an unobserved change-point time τ_{ik} is reached. After this change-point, we expect a positive slope γ_{ik} of the biomarker levels until diagnosis. This is modeled as $Y_{ijk} = \theta_{ik} + \gamma_{ik}(t_{ij} - \tau_{ik})^+ + \epsilon_{ijk}$, where $(.)^+$ denotes the positive part of the expression.

Let $\{\theta_{ik}, I_i, \log(\gamma_{ik}), \tau_{ik}\}$ be the set of subject specific parameters. The probability of a change-point in biomarker k for patient i, given all other markers, can be calculated as

$$\mathrm{P}\{I_{ik}|(I_{ik'} : k \neq k')\} = \frac{\exp\{I_{ik}F(I_{ik})\}}{1 + \exp\{F(I_{ik})\}},$$

where $F(I_{ik}) = \mu_I + \eta_I \sum_{k \neq k'} I_{ik'}$. This formulation implies that a change-point in one biomarker (e.g., $I_{ik} = 1$) increases the probability of change-points in other biomarkers whenever $\eta_I > 0$.

If $\eta_I = 0$, biomarkers are independent and the probability of a change-point for each single biomarker follows a Bernoulli distribution with parameter $1/1 + \exp(-\mu_I)$.

The mean biomarker level for patient i and biomarker k follows a Gaussian prior, $\theta_{ik} \sim N(\mu_{\theta k}, \sigma^2_{\theta k})$, where $\mu_{\theta k} \sim N(\mu_{0k}, \sigma^2_{0k})$ and the variance follows an inverse gamma distribution $\sigma^2_{\theta k} \sim \mathrm{IG}(a_{\theta k}, b_{\theta_k})$. The hyperparameters $(\mu_{0k}, \sigma^2_{0k}, a_{\theta k}, b_{\theta k})$ are assumed to be deterministic. The binary indicators $\mathbf{I}_i = \{I_{ik}\}_{k=1,\ldots,K}$ follow a Markov random field distribution, denoted as $\mathbf{I}_i \sim \mathrm{MRF}(\mu_I, \eta_I)$. Individual random effects for the rate $\log(\gamma_{ik})$ are assumed to follow independent normal distributions, $\log(\gamma_{ik}) \sim N(\mu_{\gamma k}, \sigma^2_{\gamma k})$, where $\mu_{\gamma k} \sim N(\mu_{1k}, \sigma^2_{1k})$ and $\sigma^2_{\gamma k} \sim \mathrm{IG}(a_{\gamma k}, b_{\gamma k})$. The individual change-point τ_{ik} is assumed to follow a truncated normal distribution and $\sigma^2_k \sim \mathrm{IG}(a, b)$.

The posterior distribution of all unknown parameters is approximated using a Markov chain Monte Carlo algorithm. The subset of parameters $\{\sigma^2_k,\ \mu_{\theta k},\ \sigma^2_{\theta k},\ \mu_{\gamma k},\ \sigma^2_{\gamma k}\}$ is sampled using Gibbs sampling, while $\{\mu_I,\ \eta_I\}$ are estimated via a Metropolis-Hasting algorithm.

The screening methodology for a new patient N' from a testing cohort is based on computing the posterior probability of ovarian cancer, o_i, where $o_i \in \{0, 1\}$ indicates whether patient i is a control (0) or a case (1). The posterior probability of ovarian cancer for patient N' is denoted as $\mathrm{P}(\mathrm{o}_{iN'} = 1|Y_{N'})$, where $Y_{N'}$ represents the longitudinal time series of biomarkers for patient N' up to time t_{ij}, $Y_{N'} = \{Y_{N(j',k')} : j = 1, 2, \ldots, j'; k = 1, 2, \ldots, K\}$. The posterior probability is given by

$$\mathrm{P}(\mathrm{o}_{iN'} = 1|Y_{N'}) = \frac{\mathrm{P}(Y_{N'}|\mathrm{o}_{iN'} = 1)\mathrm{P}(\mathrm{o}_{iN'} = 1)}{\sum_{i\in\{0,1\}} \mathrm{P}(Y_{N'}|\mathrm{o}_{iN'} = i)\mathrm{P}(\mathrm{o}_{iN'} = i)},$$

where $\mathrm{P}(\mathrm{o}_{N'})$ represents the prior prevalence estimated from population data, and $\mathrm{P}(Y_{N'}|\mathrm{o}_{N'})$ is estimated from the posterior predictive distribution for patient N' at each screening time using the training data from N patients.

Our findings show that the best results are obtained when combining longitudinal Cancer Antigen 125 (CA125) and Human Epididymis protein 4 (HE4), highlighting the complementary nature of the latter biomarker. This combination achieves a higher area under the receiver operating characteristic (ROC) curve and increased sensitivity at a fixed specificity, which may significantly improving early ovarian cancer detection [1].

Keywords: change-point detection; longitudinal biomarkers; ovarian cancer; CA125.

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Evaluating Wind Effects Using a Lagrangian Transport Model: Tracking Marine Debris in Northwest Spain

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Marine debris is responsible for major problems in our oceans, causing serious environmental degradation, detrimental health effects and economic losses in sectors related to the marine environment [1]. In this study, based on the results of a Lagrangian transport model, the influence of wind on debris dispersion will be addressed.

The transport and accumulation of marine debris in coastal environments are strongly influenced by hydrodynamic and atmospheric forcing. This study investigates the role of wind in the dispersion and stranding of floating debris in the Ría de Arousa (NW Spain) using the MOHID-Lagrangian model. The particle trajectories are estimated as,

$$\frac{dr_i}{dt} = v_i(r_i(t), t) + R\sqrt{\frac{2K_h}{\Delta t}}$$

where v_i represents the particle's velocity interpolated from the flow field at the particle position r_i and time instant t. In this case, as the wind effect is also considered, $v_i = v_c + v_w$, where v_c is the ocean current and v_w is the wind at $10m$. The last term introduces a random component to the particle motion, allowing for the inclusion of small-scale turbulent effects that are not resolved at the simulation's current resolution [2]. This model does also incorporate a beaching functionality, further explained in [3]. Particles frequency release depends on the rivers outflow.

We simulate the transport of particles released from the Ulla River between 2018 and 2023 for different windage factors to account for different plastic types. The analysis focuses on the spatial distribution of stranded particles along the coastline, seasonal variability in debris dispersion within the study domain, influence of the river outflow, and residence times across the estuary: internal (near the river mouth), intermediate, and external (adjacent to the open ocean).

The results reveal a clear influence of wind on debris dynamics. When wind is included in the simulations, there is a general reduction in the number of stranded particles along the coastline, as well as a decrease in the total number of floating particles within the study domain. Moreover, residence times are consistently shorter in all three zones, indicating an enhanced export of debris from the estuary. These findings suggest that wind-induced transport mechanisms play a key role in determining the fate of marine debris, influencing both its accumulation patterns and retention within coastal waters.

By examining the predominant wind regimes during different seasonal periods, this study provides insight into the relationship between wind forcing and debris dispersion, with implications for predictive modeling and potential mitigation strategies. Understanding these dynamics is crucial for developing more effective management policies aimed at reducing the impact of marine litter in estuarine and coastal environments.

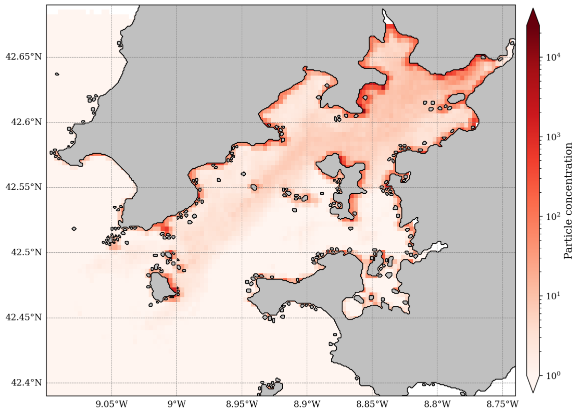

Figure 18. Particle distribution without wind.

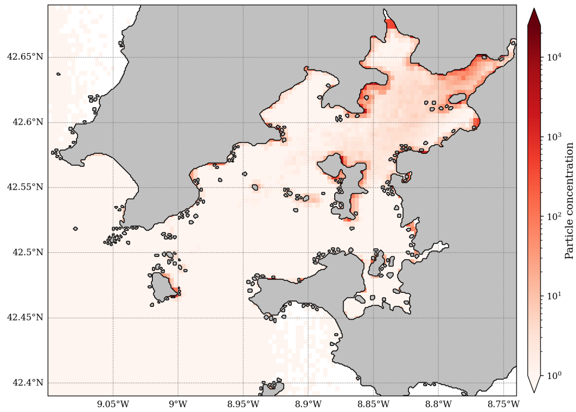

Figure 19. Particle distribution including wind.

Keywords: stability analysis; Sturm-Liouville problem; stationary solutions; nonlinear Dirac equations

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Multiphysics modeling of gas-surface interaction in a vacuum carburizing furnace using CFD and finite difference methods

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Vacuum carburizing is a heat treatment process used to enhance the surface hardness of steel components by introducing carbon into their structure. In this process, acetylene (C_2H_2) gas is injected into a vacuum furnace, where it decomposes upon contact with hot metal surfaces, releasing carbon that diffuses into the steel. This study combines Computational Fluid Dynamics (CFD) simulations in OpenFOAM with a 1D finite difference model in Python to analyze both the gas-phase transport inside the furnace and the subsequent solid-state diffusion of carbon within the steel. The CFD analysis investigates the flow dynamics of acetylene, gas dispersion, and stagnation zones, while the diffusion model evaluates carbon penetration under different exposure conditions. By integrating these approaches, this work provides insights into optimizing process parameters to improve carburizing uniformity and efficiency.

Keywords: Vacuum carburizing; Carbon diffusion; Numerical simulation.

Complex dynamics in clonal growth: Seagrass meadows as a case study

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Seagrasses are vital organisms in coastal waters, and the drastic decrease of their population in the last decades has worrying implications for marine ecosystems. Spatial models for seagrass meadows provide a mathematical framework to study their dynamical processes and emergent collective behaviour. These models are crucial to predict the response of seagrasses to different global warming scenarios, analyse the resilience of existing seagrass distributions, and optimise restoration strategies. One of the major challenges is to identify and quantify the effect of local and long-range disturbances and to characterise the seagrass response to them. Besides fundamental field measurements and observations, we present a theoretical model approach able to reproduce the observed seagrass growth dynamics and the spatial organisation resulting from the effect of different stressors. Such spatial patterns act as early signaling indicators of the proximity of tipping points that can be avoided or delayed through self-organization. We will analise the dependence of the mortality rate on local shoot densities, temperature and sulfide poisoning, that lead to multistability and excitability. Model outcomes have been successfully compared to field measurements of Posidonia oceanica meadows in the Mediterranean Sea [1, 2].

Keywords: complex systems; marine ecology; species competition; climate change.

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Positive solution for a system of integral equations via Krasnosel'skiĭ-Guo theorem

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The poster focuses on the solvability for the system of nonlinear Hammerstein integral equations. The aim is to establish sufficient conditions for the existence of positive solution for such a system. Krasnosel'skiĭ-Guo fixed point theorem and specific cone were applied in order to achieve the results. An numerical example is provided in order to illustrate the applicability of presented theorems.

Keywords: Hammerstein integral equations, Guo–Krasnosel'skiĭ theorem in cone, positive solution, Green's function.

Interplay of oscillations and alternans in a model of calcium handling in cardiac cells

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In cardiac tissue, instabilities in electrical wave propagation can lead to life-threatening arrhythmias, often associated with dysregulations in intracellular calcium handling. Calcium acts as a second messenger in cardiac cells, triggering cellular contraction. Calcium ions are primarily stored within the sarcoplasmic reticulum (SR) and are released through specialized calcium-sensitive channels called ryanodine receptors (RyR).

Two prominent forms of instability in calcium cycling are calcium alternans and calcium oscillations. In this study, we analyze these instabilities using minimal mathematical models of the calcium handling system [1, 2]. Initially, we investigate the emergence of calcium oscillations as a function of critical model parameters, including the intensity of calcium release from the SR and its subsequent uptake. Subsequently, we explore calcium alternans, characterized by fluctuations in intracellular calcium concentration that alternate between larger and smaller amplitudes on successive cardiac beats. We examine how factors such as channel inactivation, calcium diffusion, and release conductance influence the onset of alternans, and map out regions of periodic and chaotic dynamics within the model's parameter space.

Finally, we study the complex interplay between calcium oscillations and alternans, elucidating their combined behavior through mathematical bifurcation analysis. Our findings contribute to a deeper understanding of calcium-induced instabilities and their potential roles in cardiac arrhythmogenesis.

Keywords: arrhythmias, calcium dynamics, alternans, calcium oscillations.

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Coexistence of Strange Attractors for a 2-D Quadratic Family

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The aim of this poster is to show a numerical approach to the coexistence of persistent strange attractors for the quadratic two-parameter family $T_{a,b}(x, y) = (a + y^2, x + by)$.

This family $T_{a,b}$ was introduced in [1], arising as the limit family in generic unfoldings of homoclinic tangencies in the 3-D diffeomorphism scenario. Previously, in [2] and [3] the equivalent family $\widetilde{T}_{a,b}(x, y) = (ax + y, b + x^2)$ was studied as an example of endomorphisms having a critical curve separating points having two preimages from those that do not.

One of the main results in [1] is the obtaining of a region of parameters (a, b) where $T_{a,b}$ displays strange attractors. Furthermore, this region is divided into four subsets according to the kind of the obtained attractor.

The same region of parameters (a, b) is obtained in [2]. In this case, it is divided into two subsets: The first is determined by the existence of m-periodic attracting orbits ($m \leq 14$) while the other contains those parameters for which there exists a periodic attracting orbit with period greater than 14 or a chaotic attractor, though no further details are given about the class of such chaotic attractor.

Now, we provide numerical examples of the coexistence of different kinds of attractors for $T_{a,b}$: attracting periodic orbits, attracting closed curves, 1-D strange attractors and 2-D strange attractors.

In the same way that the study of family $f_a(x) = 1 - ax^2$ was crucial to proving the persistence of 1-D strange attractors, the study of family $T_{a,b}$ should be a key first step toward proving the persistence of 2-D strange attractors in a generic scenario. A first analytical approach to proving the existence of 2-D strange attractors for $T_{a,b}$ was made in [1] by considering certain values of the parameters for which invariant domains exist. In particular, for $(a, b) = (-4, -2)$, there exists a strictly invariant curvilinear triangle where $T_{-4,-2}$ is topologically conjugate to a two-dimensional tent map. This fact motivated the study of a family of piecewise expanding linear maps for which the existence of 2-D strange attractors was proved (see, for instance, [4, 5]). The extension of the results obtained for these piecewise expanding linear maps will be a first step in the study of $T_{a,b}$.

Keywords: strange attractors; chaos; quadratic maps; coexistence of attractors.

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Semilinear version of Amann's three solutions theorem

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The celebrated Amann's three solutions theorem provides a tool for studying the multiplicity of fixed points of positive operators in the normed spaces with a partial order induced by a cone. Let us recall that a cone in a normed space X is a closed, convex subset K of X such that $\alpha x \in K$ for $x \in K$ and $\alpha \geq 0$ and $K \cap (-K) = \{0\}$. A partial order in X, induced by a cone $K \subset X$, is defined as

$$x \leq y \Longleftrightarrow y - x \in K.$$

The cone K is called normal if there is a number $c > 0$ such that if $0 \leq x \leq y$ then $\|x\| \leq c\|y\|$. The cone K is said to be solid if $\operatorname{int}(K)$ is nonempty.

For $x,\, y \in X$ we shall write $x \not\leq y$ if $y - x \notin K$, $x < y$ if $x \leq y$ and $x \neq y$ and $x \ll y$ if $y - x \in \operatorname{int}(K)$. For every $x,\, y \in X$ with $x \leq y$, the set $[x, y] = \{z \in X : x \leq z \leq y\}$ is called an order interval.

Theorem (Amann's Three Solutions Theorem)[1, 2]. Let K be a normal solid cone in the normed space X, and let $u_1, v_1, u_2, v_2 \in X$ satisfy

$$u_1 < v_1 < u_2 < v_2.$$

Suppose that $A : [u_1, v_2] \to X$ is completely continuous, strongly increasing, that is, $x < y$ implies $Ax \ll Ay$. and the following inequalities hold:

$$u_1 \leq Au_1, \quad Av_1 < v_1, \quad u_2 < Au_2, \quad Av_2 \leq v_2.$$

Then A has at least three fixed point points x_1, x_2, x_3 in $[u_1, v_2]$ such that

$$u_1 \leq x_1 \ll v_1, \qquad u_2 \ll x_2 \leq v_2 \quad \text{and} \quad u_2 \not\leq x_3 \not\leq v_1.$$

We will present an extension of Amann's result to a semilinear equation $Lx = Nx$, where L is a linear Fredholm mapping of index zero and N is a nonlinear L-completely continuous operator. An application to a first order boundary value problem

$$\begin{cases} x'(t) = f(t, x(t)), \quad t \in (0, 1), \\ x(0) = \lambda[x], \end{cases}$$

will be also discussed. Here $\lambda[x] = \int_0^1 x(s)\, d\Lambda(s)$, is a bounded linear functional on $C[0, 1]$ given by a Riemann-Stieltjes integral with Λ a function of bounded variation.

This is a joint work with José Ángel Cid (Spain) and Feng Wang (China).

Keywords: normed space; cone; semilinear equation; three solutions; boundary value problem.

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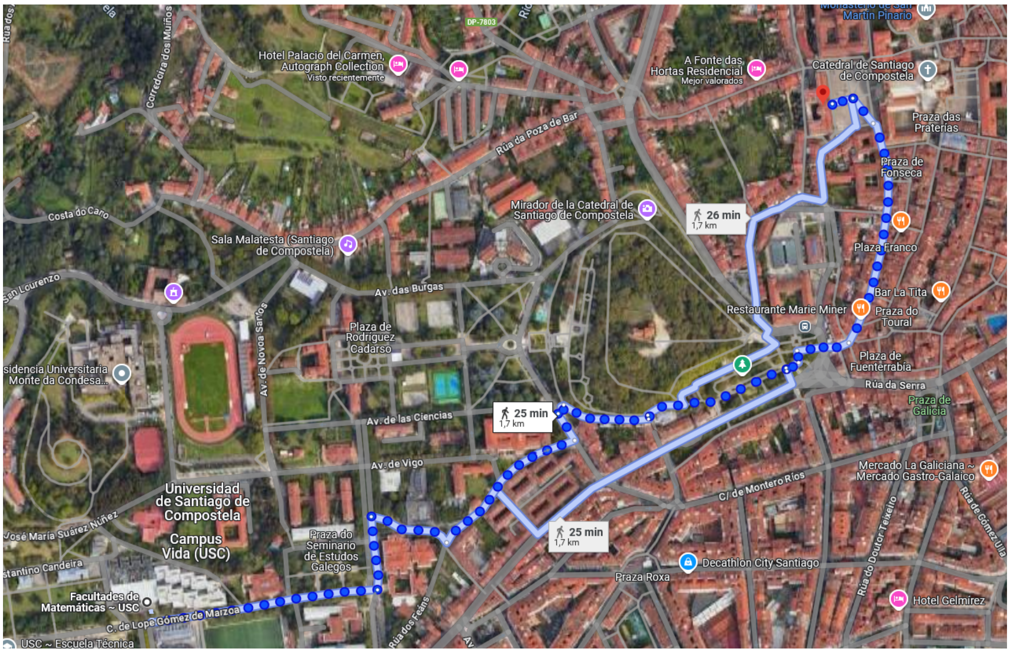

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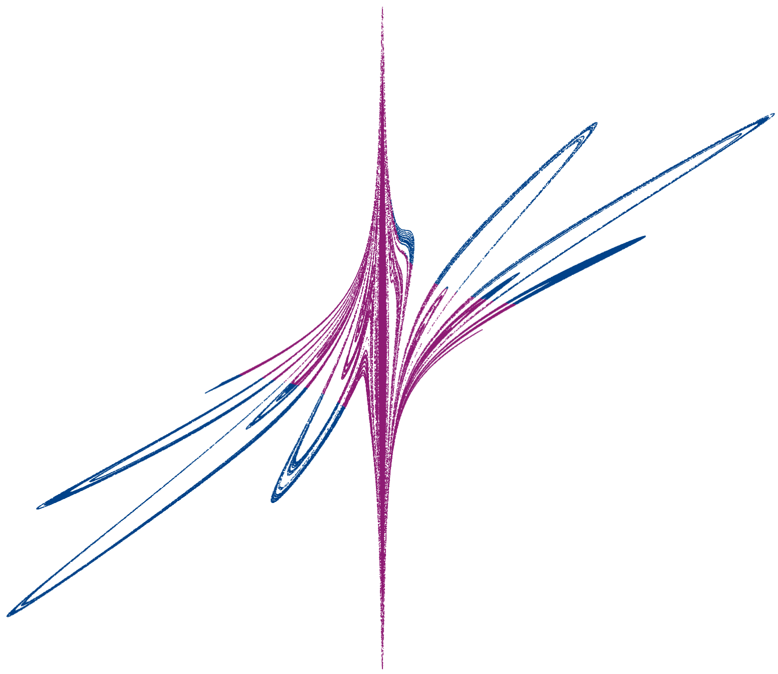

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