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Kahan-Hirota-Kimura Maps Preserving Rational Fibrations

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Abstract

The Kahan-Hirota-Kimura (KHK) maps arise from a discretization method designed for quadratic ordinary differential equations (ODEs). These maps are of significant interest in the study of integrable systems, as they frequently preserve first integrals when derived from Hamiltonian vector fields or systems with conserved quantities [1, 5]. In this work, we present a simple methodology for studying the global dynamics of planar birational maps that preserve rational (genus 0) fibrations [3] as a complement to the more geometrical approaches in [2, 4, 6]. We will revisit a particular example among the ones considered in these references, showing how KHK maps display more complex dynamical behaviors than the continuous flows they are intended to approximate.

Our analysis places special emphasis on the four cases of planar quadratic isochronous systems, denoted by S_i with $i = 1, \ldots, 4$. In particular we focus on the S_1 system. We demonstrate that the resulting one-parameter family of KHK maps, which varies with the integration step ϵ, possesses remarkable geometric properties: it preserves the original first integral and admits the continuous vector field S_1 as a Lie symmetry. A key finding is that the map inherits a discrete version of isochronicity. The dynamics on each energy level is conjugate to a rotation with an explicit rotation number function. This function depends only on the step size and not on the energy level which characterizes the invariant curve in the rational fibration induced by the first integral. Consequently, we prove that for a dense set of values ϵ, these maps are globally periodic, covering all possible periods with the exception of period 2.

In contrast, for the isochronous systems S_2, S_3 and S_4, numerical investigations suggest that the standard KHK discretizations are non-integrable, exhibiting behaviors typical of perturbed twist maps. To address this, we introduce the concept of *pseudo-KHK maps*. These alternative discretizations are constructed to ensure the preservation of first integrals and the existence of Lie symmetries.

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