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Gramian matrix characterizations in Sobolev spaces: Applications to boundary value problems and mathematical models

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Abstract

The structural theory of finitely generated shift-invariant spaces is essential for developing stable mathematical models and solving complex boundary value problems. While well-established in the classical L^2 setting, the L^2 norm fails to distinguish between smooth and noisy signals with the same energy. Consequently, applications demanding smoothness (such as medical imaging and the numerical resolution of PDEs) require the Sobolev space H^s framework. In this paper, we overcome the impracticality of existing infinite-dimensional checks by providing a highly computable characterization: a system of translates forms a Riesz basis in H^s if and only if the eigenvalues of the associated weighted Gramian matrix are uniformly bounded above and away from zero. To demonstrate the practical relevance of this finite-dimensional algebraic test, we apply it directly to boundary value problems, specifically the Poisson equation with Dirichlet boundary conditions. We show how our results guarantee well-conditioned system matrices and construct stable, convergent Petrov-Galerkin schemes. This provides a rigorous foundation for mathematical models describing physical phenomena such as heat distribution and electrostatic potentials. Furthermore, we extend this framework to mathematical models in Magnetic Resonance Imaging (MRI), proving robust image reconstruction, and we establish a necessary Nyquist-type density condition for stable non-uniform sampling in H^s.

References

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