Constant–Sign Derivatives of the Green's Function for Fourth–Order Operators

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Abstract

In this talk we present the characterization of the constant–sign behavior of the partial derivatives of the Green's function associated with the fourth–order linear differential operator

$$T[M]u(t) = u^{(4)}(t) - p\,u''(t) + Mu(t), \qquad t \in [0,1],\ p > 0,$$

subject to general two–point boundary conditions. This analysis is also fundamental for the treatment of nonlinear problems, since the detailed information on the constant–sign behavior of the Green's function and of its partial derivatives provides the structural foundations needed to define more precise and better–adapted cones in the corresponding function spaces. This, in turn, allows the rigorous application of fixed–point methods to guarantee the existence of nonlinear solutions.

Our main results provide a complete spectral description of the intervals of the parameter M for which the Green's function g_M, as well as its first derivatives with respect to the first variable, preserve a constant sign in the interior of the domain. The analysis does not rely on an explicit formula for g_M; instead, it is based on the precise identification of the boundary spaces satisfied by by the derivatives of the Green's function and on the relation of such spaces with suitable adjoint problems.

A distinctive feature of the fourth–order case is that the boundary spaces satisfied by the derivatives of the Green's function depend on certain functions arising directly from the definition of the operator and from the imposed boundary conditions. This functional dependence becomes apparent through an appropriate decomposition of the adjoint operator and leads to a more delicate and technically demanding structure, requiring refined tools to guarantee constant–sign properties.

Although the talk focuses on the fourth–order problem, we briefly discuss how this situation differs from the general n–th order problem, given by the operator

$$T_n[M]u(t) = u^{(n)}(t) + Mu(t),$$

as analyzed in [2]. In fact, the general n–th order framework can be viewed as a natural extension of the spectral approach developed in [1] for characterizing the constant sign of Green's functions under two–point boundary conditions.

In this broader setting, the boundary spaces satisfied by the derivatives of g_M depend solely on index shifts induced by differentiation, without involving additional functions or specific decompositions of the operator. Despite the fact that the analysis remains mathematically demanding, the resulting framework is more systematic and purely algebraic. The contrast with the fourth–order case highlights the structural complications introduced by the functional dependence of the boundary spaces and clarifies why more refined methods are needed when dealing with the fourth–order case.

Finally, we briefly present a numerical application of the theoretical results, computing the eigenvalues associated with the relevant boundary spaces and thus determining the intervals of M that guarantee constant–sign behavior.

References

[1] A. Cabada, L. Saavedra, *Characterization of constant sign Green's function for a two-point boundary-value problem by means of spectral theory*, Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 146, 1–95.

[2] A. Cabada, L. López-Somoza, M. Yousfi, *Spectral characterization of the constant sign derivatives of Green's function related to two point boundary value conditions*, Qualitative Theory of Dynamical Systems, 24(38) (2024), 1–57.

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