Nonlinear Differential Equations and Dynamical Systems NDEDS2026
A conference celebrating Alberto Cabada's 60-th birthday

6–10 July, 2026, Santiago de Compostela, Spain.

Applications of Amann type fixed point theorems

MIROSŁAWA ZIMA

Abstract

The celebrated Amann's three solutions theorem is a tool for studying the multiplicity of fixed points of positive operators in the normed spaces with a partial order induced by a cone. Recall that a cone in a normed space X is a closed, convex subset K of X such that $\alpha x \in K$ for $x \in K$ and $\alpha \geq 0$ and $K \cap (-K) = \{0\}$. A partial order in X, induced by a cone $K \subset X$, is defined as

$$x \leq y \Longleftrightarrow y - x \in K.$$

The cone K is called normal if there is a number $c > 0$ such that if $0 \leq x \leq y$ then $\|x\| \leq c\|y\|$. The cone K is said to be solid if int(K) is nonempty.

For $x, y \in X$ we shall write $x \not\leq y$ if $y - x \notin K$, $x < y$ if $x \leq y$ and $x \neq y$ and $x \ll y$ if $y - x \in \mathrm{int}(K)$. For every $x, y \in X$ with $x \leq y$, the set $[x, y] = \{z \in X : x \leq z \leq y\}$ is called an order interval.

Theorem (Amann's Three Solutions Theorem)[1]. Let K be a normal solid cone in the normed space X, and let $u_1, v_1, u_2, v_2 \in X$ satisfy

$$u_1 < v_1 < u_2 < v_2.$$

Suppose that $A : [u_1, v_2] \to X$ is completely continuous, strongly increasing, that is, $x < y$ implies $Ax \ll Ay$. and the following inequalities hold:

$$u_1 \leq Au_1, \;\; Av_1 < v_1, \;\; u_2 < Au_2, \;\; Av_2 \leq v_2.$$

Then A has at least three fixed point points x_1, x_2, x_3 in $[u_1, v_2]$ such that

$$u_1 \leq x_1 \ll v_1, \quad u_2 \ll x_2 \leq v_2 \quad \text{and} \quad u_2 \not\leq x_3 \not\leq v_1.$$

In first part of the talk an extension of Amann's result to a semilinear equation $Lx = Nx$ will be presented. Here L is a linear Fredholm mapping of index zero and N is a nonlinear L-completely continuous operator. An application to a first order nonlocal boundary value problem will be discussed. This is a joint work with José Ángel Cid (Spain) and Feng Wang (China) [2].

Next we will present a hybrid fixed point theorem that combines the Krasnosel'skiĭ fixed point theorem with Amann's result for the operator $T = (T_1, T_2)$. An application to boundary value problem for second order system will be given. This is a joint work with Jorge Rodríguez–López (Spain).

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Review* **18** (1976), 620–709.

[2] F. Wang, J. A. Cid, and M. Zima, Amann's three solutions theorems for semilinear operator equation, *J. Fixed Point Theory Appl.* **27:103** (2025), 20 pp.

Mirosława Zima
Institute of Mathematics, University of Rzeszów, Rzeszów, Poland
e-mail: mzima@ur.edu.pl