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An extension of Bertrand's Theorem

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In 1873 Bertrand proved the following remarkable phenomenon for planar motions under central forces: among the Newtonian systems

$$\ddot{\boldsymbol{x}} + V'(|\boldsymbol{x}|)\frac{\boldsymbol{x}}{|\boldsymbol{x}|} = 0\,, \qquad \boldsymbol{x} = \boldsymbol{x}(t) \in \mathbb{R}^2,$$

the requirement that *all bounded motions with nonzero angular momentum are periodic* implies that V has to be either the Kepler potential $V(r) = -k/r$ or the isotropic harmonic potential $V(r) = kr^2$.

The purpose of this paper is to investigate Bertrand-type phenomena for the p–Laplacian system

$$\frac{d}{dt}\big(|\dot{\boldsymbol{x}}|^{p-2}\dot{\boldsymbol{x}}\big) + V'(|\boldsymbol{x}|)\frac{\boldsymbol{x}}{|\boldsymbol{x}|} = 0\,, \qquad p > 1\,. \tag{1}$$

Our goal is to show that a *robust* Bertrand property holds only if $p = 2$ and either $V(r) = -k/r$ or $V(r) = kr^2$. To be more precise, we need the following.

Definition. *We say that V satisfies the* B-property *if the following three conditions hold:*

(i) *for every $r_0 > 0$ there exists a circular solution with radius r_0;*

(ii) *for every circular solution $\boldsymbol{x}_0$ there exists a $\delta > 0$ such that, if $\boldsymbol{x}$ is any other solution satisfying*

$$|\boldsymbol{x}(t_0) - \boldsymbol{x}_0(t_0)| + |\dot{\boldsymbol{x}}(t_0) - \dot{\boldsymbol{x}}_0(t_0)| < \delta\,,$$

for some $t_0 \in \mathbb{R}$, then $\boldsymbol{x}$ is bounded;

(iii) *every bounded solution with nonzero angular momentum is periodic.*

Our result shows that the B-property selects the Newtonian case $p = 2$ within the family of p-Laplacian dynamics. Here is the statement.

Theorem. *If the B-property holds for equation* (1)*, then necessarily $p = 2$. In such a case, either $V(r) = -k/r$, or $V(r) = kr^2$, for some positive constant k.*

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